Sone Manifold for Stationary Boltzmann Equation

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Sone Manifold

1-D Stationary Boltzmann equation Boltzmann equation:

$$\xi_1 \partial_x \mathrm{f} = \frac{1}{k} Q(\mathrm{f}, \mathrm{f}).$$

Goal:

- To construct the invariant manifolds using the Greens function approach.
- To study the coupling of Knudsen-type boundary layers and the fluid-like interior waves.
- Key: Construction of Sone Manifold.

In collaboration with Shih-Hsien Yu.

Sone Manifold

Boltzmann equation:

$$\partial_t \mathrm{f} + \boldsymbol{\xi} \cdot \partial_{\boldsymbol{x}} \mathrm{f} = \frac{1}{k} Q(\mathrm{f}, \mathrm{f}).$$

Transport: $\partial_t \mathrm{f} + \boldsymbol{\xi} \cdot \partial_{\boldsymbol{x}} \mathrm{f}$
Collision operator:
$Q(\mathrm{f}, \mathrm{f})(\boldsymbol{\xi}) \equiv \int_{\mathbb{R}^3} \int_{S_+^2} [\mathrm{f}(\boldsymbol{\xi}')\mathrm{f}(\boldsymbol{\xi}_*') - \mathrm{f}(\boldsymbol{\xi})\mathrm{f}(\boldsymbol{\xi}_*)] B(|\boldsymbol{\xi} - \boldsymbol{\xi}_*|, \theta) d\Omega d\boldsymbol{\xi}_*.$

$$\begin{cases} \boldsymbol{\xi}' = \boldsymbol{\xi} - [(\boldsymbol{\xi} - \boldsymbol{\xi}_*) \cdot \Omega]\Omega, \\ \boldsymbol{\xi}_*' = \boldsymbol{\xi}_* + [(\boldsymbol{\xi} - \boldsymbol{\xi}_*) \cdot \Omega]\Omega. \end{cases}$$

The function $B(|\boldsymbol{\xi} - \boldsymbol{\xi}_*|, \theta)$ encodes the basic physical property of the inter-molecular potential. We will consider the hard sphere models $B = |(\boldsymbol{\xi} - \boldsymbol{\xi}_*) \cdot \Omega| = |\boldsymbol{\xi} - \boldsymbol{\xi}_*| \cos \theta$.

Conservation laws for the macroscopic variables:

$$\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ \xi \\ \frac{1}{2}|\xi|^2 \end{pmatrix} [\partial_t f + \xi \cdot \partial_{\boldsymbol{x}} f] d\xi = \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ \xi \\ \frac{1}{2}|\xi|^2 \end{pmatrix} Q(f,f) d\xi = 0.$$

$$\partial_t \rho + \partial_x \cdot (\rho \boldsymbol{v}) = 0, \text{ mass},$$

$$\partial_t(\rho \boldsymbol{v}) + \partial_x \cdot (\rho \boldsymbol{v} \times \boldsymbol{v} + \boldsymbol{P}) = 0, \text{ momentum},$$

$$\partial_t(\rho E) + \partial_x \cdot (\rho \boldsymbol{v} E + \boldsymbol{P}\boldsymbol{v} + \boldsymbol{q}) = 0, \text{ energy.}$$

H-Theorem, Irreversibility $H \equiv \int_{\mathbb{R}^3} f \log f d\xi$, $\vec{H} \equiv \int_{\mathbb{R}^3} \xi f \log f d\xi$:

$$\partial_t H + \partial_{\boldsymbol{x}} \cdot \vec{H} = \frac{1}{4k} \int_{\mathbb{R}^3} \int_{S^2_+} \log \frac{ff_*}{f'f'_*} [f'f'_* - ff_*] B d\Omega d\xi_* d\xi \leq 0,$$

$= 0$ if and only if

$$\mathrm{f}(\boldsymbol{x}, t, \xi) = \frac{\rho(\boldsymbol{x}, t)}{(2\pi R\theta(\boldsymbol{x}, t))^{3/2}} e^{-\frac{|\xi - \boldsymbol{v}(\boldsymbol{x},t)|^2}{2R\theta(\boldsymbol{x},t)}} \equiv \mathrm{M}_{(\rho, \boldsymbol{v}, \theta)} \text{ Maxwellian.}$$

on 5-dimensional thermo-equilibrium manifold, $Q(f, f) = 0$:

$$\{f|\ f = \mathrm{M}_{(\rho,\boldsymbol{v},\theta)},\ \rho > 0,\ \theta > 0,\ \boldsymbol{v} \in \mathbb{R}^3\}.$$

The H-Theorem says that there is a tendency for the solution f of the Boltzmann equation to approach the equilibrium manifold.

Sone Manifold

Boltzmann equation

$$\partial_t f + \boldsymbol{\xi} \cdot \partial_{\boldsymbol{x}} f = \frac{1}{k} Q(f, f).$$

At thermo-equilibrium, $f = M$, the conservation laws become the Euler equations in gas dynamics:

$$\begin{cases} \partial_t \rho + \partial_x \cdot (\rho \boldsymbol{v}) = 0, \\ \partial_t (\rho \boldsymbol{v}) + \partial_x \cdot (\rho \boldsymbol{v} \times \boldsymbol{v} + p\boldsymbol{I}) = 0, \\ \partial_t (\rho E) + \partial_x \cdot (\rho \boldsymbol{v} E + p \boldsymbol{v}) = 0. \end{cases}$$

Fluid dynamics, thermodynamics phenomena occur around the thermo-equilibrium manifold.

Sone Manifold

To study the flows near the equilibrium manifold, linearize the Boltzmann equation around a fixed Maxwellian $\mathrm{f} = \mathrm{M} + \sqrt{\mathrm{M}}\mathrm{g}$:

$$\mathrm{g}_t + \xi \cdot \partial_{\mathbf{x}} \mathrm{g} = \mathrm{Lg},\ \text{linearized Boltzmann equation},$$

The linearized collision operator$\mathrm{Lg} = \frac{2\mathbf{Q}(\sqrt{\mathrm{M}}\mathrm{g},\mathrm{M})}{\sqrt{\mathrm{M}}}$ has kernel the tangent plane to the equilibrium manifold therough M:

$$Lg = 0 \text{ for } g \in \text{span}\{\sqrt{M},\ \xi\sqrt{M},\ |\xi|^2\sqrt{M}\}.$$

Macro projection P_0: the projection onto the kernel of L.
Micro projection: $\mathrm{P}_1 \equiv I - \mathrm{P}_0$.
Macro-Micro decomposition $\mathrm{g} = \mathrm{P}_0\mathrm{g} + \mathrm{P}_1\mathrm{g} \equiv \mathrm{g}_0 + \mathrm{g}_1$.
H-Theorem $L\mathrm{g}_1 \leq -\nu \mathrm{g}_1$ for some constant $\nu > 0$.

Sone Manifold

Linearized Euler equations, 1-dimensional,

$$(\mathrm{g}_0)_t + \mathrm{P}_0(\xi^1 \mathrm{g}_0)_x = 0.$$

Euler characteristics:

$$\begin{cases} \mathrm{P}_0\xi^1\mathrm{P}_0\mathrm{E}_i = \lambda_i\mathrm{E}_i, \ \lambda_i \text{ Euler speeds}, \ \mathrm{E}_i \text{ Euler directions}, \\ \{\lambda_1, \lambda_2, \lambda_3\} = \{-\boldsymbol{c}+u, u, \boldsymbol{c}+u\}, \ \boldsymbol{c} = \sqrt{\frac{5\theta}{3}}, \text{ (sound speed at rest)}, \end{cases}$$

Navier-Stokes equations:

$$(\mathrm{g}_0)_t + (\mathrm{P}_0\boldsymbol{\xi}\mathrm{g}_0)_x = [-kL^{-1}(\mathrm{P}_1\boldsymbol{\xi}\mathrm{g}_0)_x]_x,$$

Navier-Stokes viscosity and heat conductivity :

$$A_j = A_j(\theta) = -k\left(\mathrm{P}_1\xi^1\mathrm{E}_j, \mathrm{L}^{-1}(\mathrm{P}_1\xi^1\mathrm{E}_j)\right), \ j = 1, 2, 3.$$

1-dimensional Green's function

$$\begin{cases} (-\partial_\tau - \xi_* \partial_y - \mathrm{L})\mathbb{G}(x-y, t-\tau, \xi, \xi_*) = 0, \\ \mathbb{G}(x-y, 0, \xi, \xi_*) = \delta^1(y-x)\delta^3(\xi_* - \xi). \end{cases}$$

The Green's function contains the particle-like wave and the fluid-like waves in the Euler wave direction E_k with Navier-Stokes dissipations $A_k \equiv -(\mathrm{P}_1 \xi^1 \mathrm{E}_k, L^{-1}\mathrm{P}_1 \xi^1 \mathrm{E}_k)$:

Theorem

$$\mathbb{G}(x, t, \xi; \xi_*) = e^{-\nu(\xi_*)t}\delta(x - \xi^1 t)\delta^3(\xi - \xi_*) + \sum_{k=1}^{3} \frac{e^{-\frac{(x-\lambda_k t)^2}{4A_k(t+1)}}}{\sqrt{4A_k \pi(t+1)}} \mathrm{E}_k(\xi)\mathrm{E}_k(\xi_*) + \cdots ;$$

Invariant manifods, the linear theory

$$\xi^1\partial_x g = \frac{1}{k} Lg.$$

Definition

$$L^\infty_{\xi,3} \equiv \{h : \sup_\xi |h|(1+|\xi|)^3 < \infty\} = \boldsymbol{S} \oplus \boldsymbol{C} \oplus \boldsymbol{U}$$

is a stable-center-unstable invariant manifolds decomposition for the linear stationary Boltzmann equation $\xi^1\partial_x g = \frac{1}{k}Lg$ if

- for any given $h \in \boldsymbol{S}$, there exists a solution g to the equation for $x > 0$ with $g|_{x=0} = h$, $g(x) \to 0$ as $x \to \infty$,
- for any given $h \in \boldsymbol{U}$, there exists a solution g to the equation for $x < 0$ with $g|_{x=0} = h$, $g(x) \to 0$ as $x \to -\infty$,
- any given $h \in \boldsymbol{C}$ is a constant solution of the equation.

Sone Manifold

We will start with the time-dependent equation

$$\partial_t g + \xi^1 \partial_x g = \frac{1}{k} L g, \;\; x > 0,$$

and use the time-asymptotic analysis to construct the stationary solution:

$$\xi^1 \partial_x g = \frac{1}{k} L g, \;\; x > 0.$$

To obtain time-asymptotic convergence, we need:

- Time-asymptotic compactness.
- Suitable boundary flux at $x = 0, \pm\infty$.
- These require pointwise control of the time dependent solutions, thereby the Green's function approach.

Sone Manifold

Euler projections $\mathrm{B}_j,\ j = 1, 2, 3,$

$$\mathrm{B}_j\mathrm{k} \equiv (\mathrm{E}_j, \mathrm{k})\mathrm{E}_j,\ \mathrm{B}_+ \equiv \sum_{\lambda_k > 0} \mathrm{B}_k,\ \text{upwind Euler projections},$$

Euler Flux Projections $\tilde{\mathrm{B}}_j,\ j = 1, 2, 3,$

$$\tilde{\mathrm{B}}_j\mathrm{g} \equiv \frac{(\mathrm{E}_j, \xi^1\mathrm{g})\mathrm{E}_j}{\lambda_j},\ \tilde{\mathrm{B}}_+ \equiv \sum_{\lambda_k > 0} \tilde{\mathrm{B}}_k,\ \text{upwind Euler flux projections.}$$

Sone Manifold

Theorem

If $\lambda_j,\ j = 1, 2, 3$, are non-zero, then for any $\mathrm{h} \in L^\infty_{\xi,3}$, $\mathrm{g}(x) = \mathbb{S}_x\mathrm{h}$ *and* $\mathrm{g}(x) = \mathbb{U}_x\mathrm{h}$ *given below are solution of* $\xi^1\partial_x\mathrm{g} = \frac{1}{k}\mathrm{Lg}$:

Stable steady linear Boltzmann flows

$$\begin{cases} \mathbb{S}_x\mathrm{h} \equiv \int_0^\infty \mathbb{G}(x,s)\xi^1(1-\tilde{\mathrm{B}}_+)\mathrm{h}ds, \\ \mathbb{S}_x\mathrm{h} = O(1)e^{-\alpha|x|},\ x \to \infty \end{cases}$$

Unstable steady linear Boltzmann flows

$$\begin{cases} \mathbb{U}_x\mathrm{h} \equiv -\int_0^\infty \mathbb{G}(x,s)\xi^1(1-\tilde{\mathrm{B}}_-)\mathrm{h}ds, \\ \mathbb{U}_x\mathrm{h} = O(1)e^{-\alpha|x|},\ x \to -\infty \end{cases}$$

Linear Stable-Center-Unstable Decomposition

$$L^\infty_{\xi,3} = \boldsymbol{S} \oplus \boldsymbol{C} \oplus \boldsymbol{U},\ \boldsymbol{S} \equiv \mathbb{S}_{0+}(L^\infty_{\xi,3}),\ \boldsymbol{C} \equiv \tilde{\mathrm{P}}_0(L^\infty_{\xi,3}),\ \boldsymbol{U} \equiv \mathbb{U}_{0-}(L^\infty_{\xi,3})$$

Lemma (Stable manifold)

For any $\mathrm{b} \in L^\infty_{\xi,3}$, $\mathrm{g} \equiv \mathbb{S}\mathrm{b}$ *solves the linear stationary Boltzmann equation for* $x > 0$ *and* $\mathrm{g} \to 0$ *as* $x \to \infty$.

The source has no upwind Euler component$\mathrm{B}_+\xi^1(\mathrm{b} - \tilde{\mathrm{B}}_+\mathrm{b}) = 0$, and so the convolution with the Green's function satisfies:

$$\|\mathrm{g}(x)\|_{L^\infty_{\xi,3}} = \left\| \lim_{t\to\infty} \int_0^t \mathbb{G}(x, t-\tau)[\xi^1(\mathrm{b} - \tilde{\mathrm{B}}_+\mathrm{b})]d\tau \right\|_{L^\infty_{\xi,3}}$$
$$\leq \lim_{t\to\infty} O(1) \int_0^t \left(\sum_{\lambda_j>0} \frac{e^{-\frac{|x-\lambda_j(t-\tau)|^2}{C(t-\tau)}}}{(t-\tau+1)} + \sum_{\lambda_j<0} \frac{e^{-\frac{|x-\lambda_j(t-\tau)|^2}{C(t-\tau)}}}{\sqrt{(t-\tau+1)}} \right)$$
$$\cdot \left\| (1+|\xi^1|)\mathrm{b} \right\|_{L^\infty_{\xi,3}} d\tau = \left(\frac{O(1)}{\sqrt{1+x}} + e^{-x/C} \right) \|\mathrm{b}\|_{L^\infty_{\xi,3}}.$$

Lemma (Gauss Lemma)

For any b $\in L^\infty_{\xi,3}$,

$$\mathrm{b} = \int_0^\infty \mathbb{G}(0+,\tau)[\xi^1 \mathrm{b}]d\tau - \int_0^\infty \mathbb{G}(0-,\tau)[\xi^1 \mathrm{b}]d\tau.$$

The only distribution in $\mathbb{G}$ is $h^0 = \delta(x-\xi^1 t)\delta^3(\xi-\xi_*)e^{-\nu(\xi_*)t}$:

$$\int_0^\infty \mathbb{G}(0+,\tau)[\xi^1 \mathrm{b}]d\tau - \int_0^\infty \mathbb{G}(0-,\tau)[\xi^1 \mathrm{b}]d\tau$$

$$= \int_0^\infty \mathrm{h}^0(0+,\tau)[\xi^1 \mathrm{b}_+]d\tau - \int_0^\infty \mathrm{h}^0(0-,\tau)[\xi^1 \mathrm{b}_-]d\tau = \mathrm{b}_+ + \mathrm{b}_- = \mathrm{b},$$

where $\mathrm{b}_\pm$:

$$\begin{cases} \mathrm{b}_+(\xi) = \mathrm{b}(\xi) \text{ for } \xi^1 > 0, \\ 0 \text{ else,} \end{cases} \quad \mathrm{b}_-(\xi) = \begin{cases} \mathrm{b}(\xi) \text{ for } \xi^1 < 0, \\ 0 \text{ else.} \end{cases}$$

Lemma (Invariant manifolds decomposition)

For any b $\in L^\infty_{\xi,3}$,

$$\mathsf{b} = \tilde{\mathsf{P}}_0\mathsf{b} + \mathbb{S}_{0+}\mathsf{b} + \mathbb{U}_{0-}\mathsf{b},$$
$$\mathbb{S}_{0+}\tilde{\mathsf{B}}_- = \mathbb{U}_{0-}\tilde{\mathsf{B}}_+ = 0,$$
$$\mathbb{S}_{0+}\mathbb{U}_{0-}\mathsf{b} = \mathbb{U}_{0-}\mathbb{S}_{0+}\mathsf{b} = 0.$$

$$\mathsf{b} = \int_0^\infty \mathbb{G}(0+,\tau)[\xi^1\mathsf{b}]d\tau - \int_0^\infty \mathbb{G}(0-,\tau)[\xi^1\mathsf{b}]d\tau \text{ [Gauss]}$$
$$= \int_0^\infty \mathbb{G}(0+,\tau)[\xi^1(1-\tilde{\mathsf{B}}_+)\mathsf{b}]d\tau + \int_0^\infty \mathbb{G}(0+,\tau)[\xi^1\tilde{\mathsf{B}}_+\mathsf{b}]d\tau$$
$$-\int_0^\infty \mathbb{G}(0-,\tau)[\xi^1(1-\tilde{\mathsf{B}}_-)\mathsf{b}]d\tau - \int_0^\infty \mathbb{G}(0-,\tau)[\xi^1\tilde{\mathsf{B}}_-\mathsf{b}]d\tau \text{ [definition]}$$
$$= \mathbb{S}_{0+}\mathsf{b} + \tilde{\mathsf{B}}_+\mathsf{b} + \mathbb{U}_{0-}\mathsf{b} + \tilde{\mathsf{B}}_-\mathsf{b} = \tilde{\mathsf{P}}_0\mathsf{b} + \mathbb{S}_{0+}\mathsf{b} + \mathbb{U}_{0-}\mathsf{b}.$$

Sone Manifold

Lemma (Exponential decay)

For some positive constant $\alpha = O(1)k \min\{|\lambda_j|,\ j = 1, 2, 3, \}$ *the stationary solution* $\mathrm{g} \equiv \mathbb{S}\mathrm{b}$ *satisfies*

$$\|\mathrm{g}(x)\|_{L^\infty_{\xi,3}} = O(1)e^{-\alpha|x|}\,\|\mathrm{b}\|_{L^\infty_{\xi,3}}\,,\ x \to \infty.$$

Similarly for $\mathrm{g} \equiv \mathbb{U}\mathrm{b}$ *satisfies*

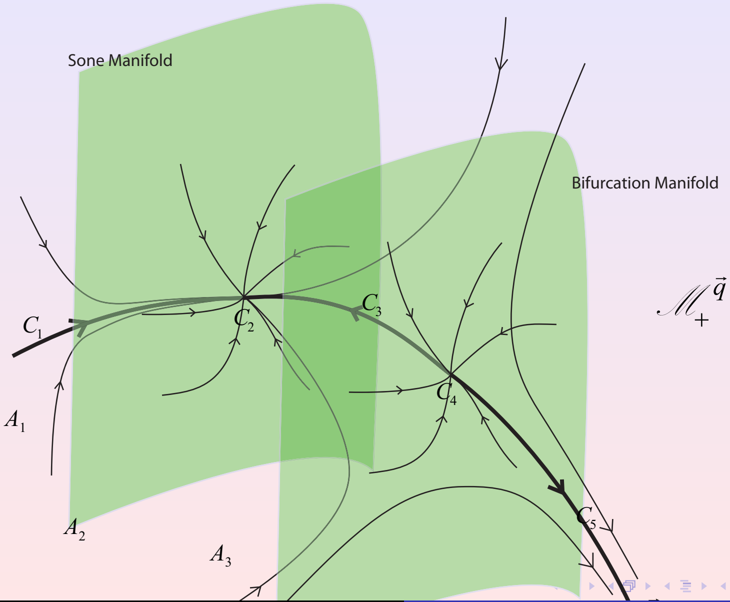
$$\|\mathrm{g}(x)\|_{L^\infty_{\xi,3}} = O(1)e^{-\alpha|x|}\,\|\mathrm{b}\|_{L^\infty_{\xi,3}}\,,\ x \to -\infty.$$

Proof.

The estimate from Green's function pointwise estimates yield algebraic rate. The exponential rate is proved by the weighted energy method. □

Sone Manifold

- The boundary layers thus obtained are of the Knudsen-type.
- The linear theory can be generalized to the nonlinear theory using standard techniques in the dynamical system because of the spectral gap $\alpha > 0$. This is a weakly nonlinear theory, with the strength of the nonlinearity of the order of α.
- The spectral gap vanishes, $\alpha = O(1)k \min\{|\lambda_j|,\ j = 1, 2, 3, \} \to 0$, as some $\lambda_j \to 0$.
- Physically, when one of the Euler characteristics is near zero, there is the coupling of the boundary Knudsen-type wave with one of the fluid-like waves. A strong nonlinear theory is needed here.
- the explicit construction of the Sone Manifold is the essential step.


Sone Manifold
Bifurcation Manifold
C_1
C_2
C_3
C_4
C_5
A_1
A_2
A_3
$\mathscr{M}_+^{\vec{q}}$

Construction of Sone Manifold

Step I. Subsonic condensation

- The Sone Manifold consists of Knudsen-type boundary waves for supersonic condensation.
- Consider small $\lambda_3 = \epsilon$, transonic condensation.
- Slowly decaying solution

$$\begin{cases} \psi(x) = \phi^\epsilon e^{\eta(\epsilon)x}, \ \eta(\epsilon) = O(1)\epsilon, \\ \frac{1}{\xi^1} \mathrm{L}^\epsilon \phi = \eta\phi. \end{cases}$$

- Uniformly bounded operator for subsonic condensation, $\epsilon > 0$

$$\begin{cases} \mathrm{B}_3^{\sharp,\epsilon}\mathrm{f}_0 \equiv \dfrac{(\mathrm{E}_3^\epsilon, \xi^1 \mathrm{f}_0)}{(\mathrm{E}_3^\epsilon, \xi^1 \ell_3^\epsilon)} \ell_3^\epsilon, \quad \ell_3^\epsilon \equiv \dfrac{\phi^\epsilon - \mathrm{E}_3^\epsilon}{\epsilon}, \\ \mathbb{S}_x^{\sharp,\epsilon}\mathrm{f}_0 \equiv \displaystyle\int_0^\infty \mathbb{G}^\epsilon(x,\tau)[\xi^1(1 - \tilde{\mathrm{B}}_1^\epsilon - \tilde{\mathrm{B}}_2^\epsilon - \mathrm{B}_3^{\sharp,\epsilon})\mathrm{f}_0]d\tau, \ x > 0, \\ \mathbb{S}_x^{\sharp,\epsilon} = O(1)e^{-\alpha x}, \ \text{for some } \alpha > 0 \text{ independent of } \epsilon. \end{cases}$$

Step I. Subsonic condensation

Uniformly bounded operator for subsonic condensation, $\epsilon < 0$

$$\begin{cases} \mathbb{U}_x^{\sharp,\epsilon} \mathrm{f}_0 \equiv -\int_0^\infty \mathbb{G}^\epsilon(x,\tau)[\xi^1(1-\mathrm{B}_3^{\sharp,\epsilon})\mathrm{f}_0]d\tau, \; x<0, \\ \mathbb{U}_x^{\sharp,\epsilon} = O(1)e^{-\alpha|x|}, \; \text{for some } \alpha>0 \text{ independent of } \epsilon. \end{cases}$$

Uniformly bounded operator for subsonic condensation, $\epsilon > 0$

$$\begin{cases} \mathbb{S}_x^{\sharp,\epsilon} \mathrm{f}_0 \equiv \int_0^\infty \mathbb{G}^\epsilon(x,\tau)[\xi^1(1-\tilde{\mathrm{B}}_1^\epsilon-\tilde{\mathrm{B}}_2^\epsilon-\mathrm{B}_3^{\sharp,\epsilon})\mathrm{f}_0]d\tau, \; x>0, \\ \mathbb{S}_x^{\sharp,\epsilon} = O(1)e^{-\alpha x}, \; \text{for some } \alpha>0 \text{ independent of } \epsilon. \end{cases}$$

Step II. Approximate Knudsen operator.

- To construct Knudsen operator $\mathbb{S}_x^{\flat,-\epsilon}$ for supersonic condensation, $-\epsilon < 0$.
- Conjugate operator

$$\bar{\mathbb{S}}_x^{\sharp,\epsilon}h \equiv \sqrt{\frac{M_\epsilon}{M_{-\epsilon}}}\mathbb{S}_x^{\sharp,\epsilon}\left(\sqrt{\frac{M_{-\epsilon}}{M_\epsilon}}h\right), x > 0,$$

  as an accurate approximation for the Knudsen operator.
- This allows for an iteration scheme for the construction of the exact Knudsen operator.

Step III. Exact Knudsen operator.

Lemma

Knudsen operator $\mathbb{S}_x^{\flat,-\epsilon},\ -\epsilon<0:$

There exist unique bounded operators $\mathbb{S}_x^{\flat,-\epsilon}$, $x>0$, *and* $\Lambda^{-\epsilon}$ *on Range*$(\mathbb{S}_{0+}^{\sharp,-\epsilon})$ *satisfying, for any* $\mathrm{b}\in$ *Range*$(\mathbb{S}_{0+}^{\sharp,-\epsilon})$,

$$\begin{cases} \mathrm{b}=\mathbb{S}_{0+}^{\flat,-\epsilon}\mathrm{b}+\Lambda^{-\epsilon}(\mathrm{b})\phi^{-\epsilon},\ \Lambda^{-\epsilon}(\mathrm{b})\in\mathbb{R}, \\ \|\mathbb{S}_x^{\flat,-\epsilon}\mathrm{b}\|_{L^\infty_{\xi,3}}\le O(1)\|\mathrm{b}\|_{L^\infty_{\xi,3}}e^{-\alpha x},\ x>0,\ \ \alpha \text{ independent of } \epsilon, \\ \|\mathbb{S}_{0+}^{\flat,-\epsilon}\mathrm{b}-\bar{\mathbb{S}}_{0+}^{\sharp,\epsilon}\mathrm{b}\|_{L^\infty_{\xi,3}}\le O(1)|\log\epsilon|\epsilon\|\mathrm{b}\|_{L^\infty_{\xi,3}}, \\ (\xi^1\partial_x-\mathrm{L}^{-\epsilon})\mathbb{S}_x^{\flat,-\epsilon}\mathrm{b}=0,\ x>0. \end{cases}$$

Proof.

Iterations from the conjugate operator $\bar{\mathbb{S}}_x^{\sharp,\epsilon}$. □

Discontinuity of linear operators

$$\begin{cases} \mathring{\mathbb{S}}_{0+}\mathrm{b} \equiv \lim\limits_{\epsilon\to 0+} \mathbb{S}^{\sharp,\epsilon}_{0+}\mathrm{b},\ \mathring{\mathbb{S}}_x\mathrm{b} \equiv \lim\limits_{\epsilon\to 0+} \mathbb{S}^{\sharp,\epsilon}_x\mathrm{b},\ x>0;\ \text{ Stable flow}, \\ \mathring{\mathbb{U}}_{0-}\mathrm{b} \equiv \lim\limits_{\epsilon\to 0-} \mathbb{U}^{\sharp,\epsilon}_{0-}\mathrm{b},\ \mathring{\mathbb{U}}_x\mathrm{b} \equiv \lim\limits_{\epsilon\to 0-} \mathbb{U}^{\sharp,\epsilon}_x\mathrm{b},\ x<0;\ \text{ Unstable flow}, \\ \mathring{\ell}_3 \equiv \lim\limits_{\epsilon\to 0} \ell^{\epsilon}_3: \\ \mathring{\mathbb{C}}_0\mathrm{b} = \sum\limits_{j=1}^{2} \tilde{\mathrm{B}}^0_j\mathrm{b} + \mathrm{B}^0_3\mathrm{b} + \dfrac{(\xi^1\mathrm{E}^0_3, \mathrm{P}^0_1\mathrm{b})}{(\xi^1\mathrm{E}^0_3, \mathring{\ell}_3)}\mathring{\ell}_3 - \mathrm{B}^0_3(\mathring{\mathbb{S}}_{0+} + \mathring{\mathbb{U}}_{0-})\mathrm{P}^0_1\mathrm{b},\ \text{ Center} \end{cases}$$

$$\begin{cases} \lim\limits_{\epsilon\to 0-} \mathit{Range}(\mathbb{S}^{\sharp,\epsilon}_{0+}) = \mathit{Range}(\mathring{\mathbb{S}}_{0+}) \oplus \mathit{span}(\mathrm{E}^0_3), \\ \lim\limits_{\epsilon\to 0+} \mathit{Range}(\mathbb{S}^{\sharp,\epsilon}_{0+}) = \mathit{Range}(\mathring{\mathbb{S}}_{0+}), \\ \lim\limits_{\epsilon\to 0+} \mathit{Range}(\mathbb{U}^{\sharp,\epsilon}_{0-}) = \mathit{Range}(\mathring{\mathbb{U}}_{0-}) \oplus \mathit{span}(\mathrm{E}^0_3), \\ \lim\limits_{\epsilon\to 0-} \mathit{Range}(\mathbb{U}^{\sharp,\epsilon}_{0-}) = \mathit{Range}(\mathring{\mathbb{U}}_{0-}). \end{cases}$$

Sone Manifold

Nonlinear invariant manifolds M_u^ϵ, M_+^ϵ, $(\epsilon < 0)$; M_s^ϵ, M_-^ϵ, $(\epsilon > 0)$: the nonlinear unstable manifold, the center-stable manifold, (supersonic); the nonlinear stable manifold, the nonlinear center-unstable, (subsonic) defined as graphs

$$\begin{cases} F_u^\epsilon : \mathit{Range}(\mathbb{U}_{0-}^{\sharp,\epsilon}) \longmapsto \mathit{Range}\left(\sum_{j=1}^{2} \tilde{B}_j^\epsilon + B_3^{\sharp,\epsilon} + \mathbb{S}_{0+}^{\sharp,\epsilon}\right), \ \epsilon < 0 \\ F_s^\epsilon : \mathit{Range}(\mathbb{S}_{0+}^{\sharp,\epsilon}) \longmapsto \mathit{Range}\left(\mathbb{U}_{0-}^{\sharp,\epsilon} + \sum_{j=1}^{2} \tilde{B}_j^\epsilon + B_3^{\sharp,\epsilon}\right), \ \epsilon > 0, \\ F_-^\epsilon : \mathit{Range}\left(\mathbb{U}_{0-}^{\sharp,\epsilon} + \sum_{j=1}^{2} \tilde{B}_j^\epsilon + B_3^{\sharp,\epsilon}\right) \longmapsto \mathit{Range}\left(\mathbb{S}_{0+}^{\sharp,\epsilon}\right), \ \epsilon > 0, \\ F_+^\epsilon : \mathit{Range}\left(\sum_{j=1}^{2} \tilde{B}_j^\epsilon + B_3^{\sharp,\epsilon} + \mathbb{S}_{0+}^{\sharp,\epsilon}\right) \longmapsto \mathit{Range}\left(\mathbb{U}_{0-}^{\sharp,\epsilon}\right), \epsilon < 0. \end{cases}$$

Sone Manifold

Center manifold=intersection of center-stable with center-unstable manifolds. With local Maxwellian coordinates, the flows on center manifold are governed by Burgers type equations.

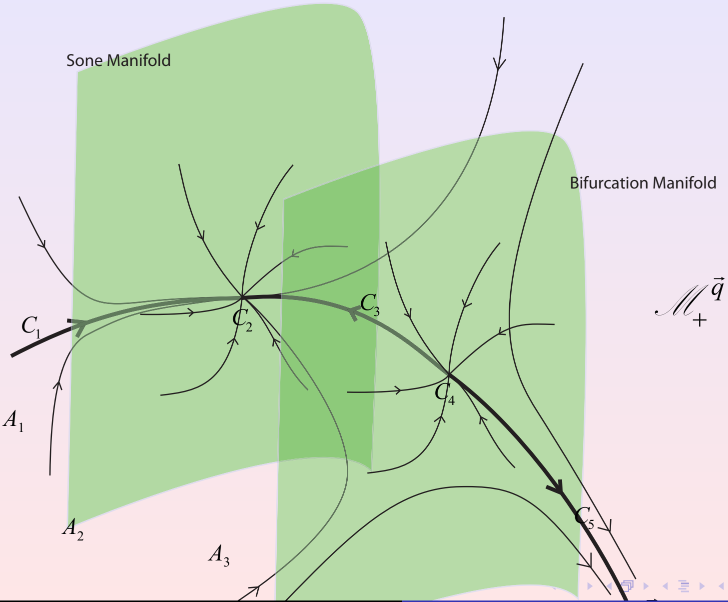
Bifurcation manifold = nonlinear manifold based on subsonic condensation stable operator $\mathbb{S}_x^{\sharp,\epsilon},\ x > 0,\ \epsilon > 0,$

Sone manifold = nonlinear manifold based on supersonic Knudsen operator $\mathbb{S}_x^{\flat,\epsilon},\ x > 0,\ \epsilon < 0.$

Two-scale flows: Fast, Knudsen type flows on Bifurcation and Sone manifolds; slow fluid-like flows on center manifold; two scale flows in general.

Monotonicity of Boltzmann shock profiles due to Burgers type dynamics on the center manifold.

Sone Manifold



Sone Manifold

Bifurcation phenomena.

The flux is conserved for steady flows

$$(\Phi_i, \xi^1 f)_x = (\Phi, \frac{1}{\kappa} Q(f,f)) = 0,\ Phi_i,\ i = 1,2,3, \text{collision invariant.}$$

The invariant manifolds depend smoothly on the flux. For the Euler equations,

$$\vec{U}_t + \vec{F}(\vec{U}) = 0$$

a perturbation in the characteristic direction

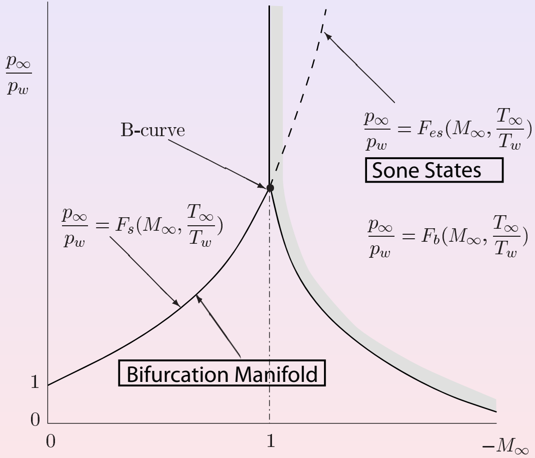
$$\vec{U}_2 = \vec{U}_1 + \epsilon \vec{r}_i(\vec{U}_1) + O(1)\epsilon^2,$$

with the resonance case $\lambda = O(1)\epsilon$, the flux changes little:

$$\vec{F}(\vec{U}_2) - \vec{F}(\vec{U}_1) = \epsilon \lambda_i(\vec{U}_1) + O(1)\epsilon^2 = O(1)\epsilon^2.$$

Thus a small change $O(1)\epsilon^2$ of the flux can induce a relatively large change ϵ of the states. This implies the large changes of Sone and Bifurcation manifolds in the transonic condensation case, for instance. And the bifurcation phenomena occur.

Sone Manifold



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