Some critical phenomena in the Cucker-Smale model

Seung Yeal Ha

Department of Mathematical Sciences
Seoul National University

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Outline

What is flocking ?

- Definition

An interacting N mechanical particle system $\{(x_i(t), v_i(t))\}_{i=1}^N$ exhibits asymptotic (velocity) flocking $\Longleftrightarrow$

1. Formation of a group.

$$\sup_{0\le t<\infty} |x_i(t) - x_j(t)| < \infty, \quad \forall\ i \neq j.$$

2. Velocity alignment.

$$\lim_{t\to\infty} |v_i(t) - v_j(t)| = 0, \quad \forall\ i \neq j.$$

The Cucker-Smale model (2007)

- Emergent behavior in flocks: IEEE Trans. Automat. Control (2007):

(x_i, v_i) position and velocity of i-th agent

$$\frac{dx_i}{dt} = v_i, \qquad 1\frac{dv_i}{dt} = \frac{K}{N}\sum_{j=1}^{N} \psi_{cs}(|x_j - x_i|)(v_j - v_i).$$

where ψ is a communication rate (modeling issue):

$$\psi_{cs}(|x_i - x_j|) = \frac{1}{(1 + |x_i - x_j|^2)^\beta} \geq 0, \qquad \beta \geq 0.$$

- Attractive forcing is built in the C-S model : In $d = 1$

$$v_j > v_i \quad \Longrightarrow \quad \psi_{cs}(|x_j - x_i|)(v_j - v_i) > 0 \ : \ \text{acceleration of } v_i,$$
$$v_j < v_i \quad \Longrightarrow \quad \psi_{cs}(|x_j - x_i|)(v_j - v_i) < 0 \ : \ \text{deceleration of } v_i,$$

$\diamond$ References: Cucker-Smale '07, J. Shen '07, Ha-Tadmor '08, Ha-Liu '09, Ha-Lee-Levy '09, Duan-Fornasier-Toscani '10, Carrillo-Fornasier-Rosado-Toscani '10, Ahn-Ha '10, Bolley-Canizo-Carrillo '10, Motsch-Tadmor '10, Ha-Bruno-Lattanzio-Slemrod, '12, Choi-H '14, $\cdots$

Some properties of the C-S model

$$\frac{dx_i}{dt} = v_i, \qquad \frac{dv_i}{dt} = \frac{K}{N}\sum_{j=1}^{N} \psi_{cs}(|x_j - x_i|)(v_j - v_i).$$

- Translation invariance under the transformation:

$$(x_i, v_i) \quad \Longrightarrow \quad (x_i + ct, v_i + c).$$

- Traveling wave solution (flocking solution, relative equilibrium)

$$v_i = v^\infty, \quad x_i = x_{i0} + v^\infty t, \qquad i = 1, \cdots, N.$$

WLOG, we assume

$$\sum_{i=1}^{N} x_i(t) = 0, \qquad \sum_{i=1}^{N} v_i(t) = 0.$$

Why Cucker-Smale model ?

- **Connection to hyperbolic conservation laws**:

"Pressureless gas-dynamics with a nonlocal source terms"

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \quad (x, v) \in R^d \times R^d,$$
$$\partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u)$$
$$= -K \int_{R^d} \psi(|x-y|)(u(y) - u(x))\rho(x)\rho(y)dy.$$

cf. Kinetic C-S model (after mean-field limit)

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (L(f)f) = 0.$$

Moment system, mono-kinetic ansatz

- **Connection to statistical mechanics**:

The Kuramoto model for synchronization (statistical mechanics)

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad \text{i.e.,}$$

After one differentiation

$$\begin{aligned} \dot{\theta}_i &= \omega_i, \\ \dot{\omega}_i &= \frac{K}{N}\sum_{j=1}^{N}\cos(\theta_j - \theta_i)(\omega_j - \omega_i). \end{aligned}$$

$$(\theta_i, \omega_i) \quad \Longleftrightarrow \quad (x_i, v_i).$$

Relation to Rational Thermodynamics:

Consider a mixture of fluids consisting of n-constituents (Ruggeri-Simic '07).

$$
\begin{cases}
\dfrac{\partial \rho_\alpha}{\partial t} + \mathrm{div}\,(\rho_\alpha \mathbf{v}_\alpha) = \tau_\alpha, \\[2ex]
\dfrac{\partial(\rho_\alpha \mathbf{v}_\alpha)}{\partial t} + \mathrm{div}\,(\rho_\alpha \mathbf{v}_\alpha \otimes \mathbf{v}_\alpha - \mathbf{t}_\alpha) = \mathbf{m}_\alpha, \qquad (\alpha = 1, 2, \ldots, n) \\[2ex]
\dfrac{\partial \left(\frac{1}{2}\rho_\alpha v_\alpha^2 + \rho_\alpha \varepsilon_\alpha\right)}{\partial t} + \mathrm{div}\left\{ \left(\dfrac{1}{2}\rho_\alpha v_\alpha^2 + \rho_\alpha \varepsilon_\alpha\right)\mathbf{v}_\alpha - \mathbf{t}_\alpha \mathbf{v}_\alpha + \mathbf{q}_\alpha \right\} = e_\alpha.
\end{cases}
$$

Assume that

$$
\sum_{\alpha=1}^{n} \tau_\alpha = 0, \quad \sum_{\alpha=1}^{n} \mathbf{m}_\alpha = \mathbf{0}, \quad \sum_{\alpha=1}^{n} e_\alpha = 0.
$$

Consider a spatially homogeneous mixture with $\tau_\alpha = 0$:

$$\frac{d\rho}{dt} = 0, \quad \frac{d\rho\mathbf{v}}{dt} = 0, \quad \frac{d}{dt}\left(\rho\varepsilon + \frac{1}{2}\rho v^2\right) = 0$$

$$\frac{d\rho_i}{dt} = 0$$

$$\frac{d\rho_i\mathbf{v}_i}{dt} = -\sum_{k=1}^{n-1}\psi_{ik}\Big(\frac{\mathbf{u}_k}{T_k} - \frac{\mathbf{u}_n}{T_n}\Big), \tag{1}$$

$$\frac{d}{dt}\left(\rho_i\varepsilon_i + \frac{1}{2}\rho_i u_i^2\right) = -\sum_{k=1}^{n-1}\theta_{ik}\Big(\frac{1}{T_n} - \frac{1}{T_k}\Big),$$

where

$$\rho = \sum_{\alpha=1}^{n}\rho_\alpha, \qquad \rho\mathbf{v} = \sum_{\alpha=1}^{n}\rho_\alpha\mathbf{v}_\alpha$$

$$\rho\varepsilon = \sum_{\alpha=1}^{n}\left(\rho_\alpha\varepsilon_\alpha + \frac{1}{2}\rho_\alpha u_\alpha^2\right), \qquad \mathbf{u}_\alpha = \mathbf{v}_\alpha - \mathbf{v}, \tag{2}$$

Assume that

$$\mathbf{v} = 0, \quad \rho_\alpha = 1, \quad \varepsilon_\alpha = T_\alpha,$$

We assume also that the velocity vector is one-dimensional $\mathbf{v}_\alpha = (v_\alpha, 0, 0)$ and in this way, the system reduces to:

$$\frac{dv_i}{dt} = -\sum_{k=1}^{n-1} \psi_{ik} \left(\frac{v_k}{T_k} - \frac{v_n}{T_n} \right), \qquad i = 1, 2, \dots n-1,$$

$$\frac{d}{dt}\left(T_i + \frac{1}{2} v_i^2 \right) = -\sum_{k=1}^{n-1} \theta_{ik} \left(\frac{1}{T_n} - \frac{1}{T_k} \right),$$

$$v_n = -\sum_{i=1}^{n-1} v_i, \qquad T_n = n\bar{T}_0 - \sum_{i=1}^{n-1} T_i - \frac{1}{2} \sum_{\alpha=1}^{n} v_\alpha^2.$$

By some transformation, we can change the previous model to the generalized C-S model (H-Ruggeri '16):

$$\frac{dx_\alpha}{dt} = v_\alpha, \quad t > 0, \quad \alpha = 1, \cdots, n,$$

$$\frac{dv_\alpha}{dt} = \sum_{\beta=1}^{n} \phi_{\alpha\beta}\Big(\frac{v_\beta}{T_\beta} - \frac{v_\alpha}{T_\alpha}\Big),$$

$$\frac{d}{dt}\left(T_\alpha + \frac{1}{2}v_\alpha^2\right) = \sum_{\beta=1}^{n} \zeta_{\alpha\beta}\Big(\frac{1}{T_\alpha} - \frac{1}{T_\beta}\Big).$$

The classical C-S model arise from the part of a bigger system in rational thermodynamics

What I plan to discuss today

- Critical phenomena

  1. From local flocking to global flocking in deterministic setting
  2. From infinite variance to zero variance in random environment

Existence of positive critical coupling strength

This is a joint work with D. Ko, X. Zhang and Y. Zhang

Global flocking problem

- **Definition**: For an mechanical interacting system $\mathcal{P} := \{(x_i, v_i)\}_{i=1}^N$, $\mathcal{P}$ exhibits asymptotic (global, mono-cluster) flocking if and only if the following two conditions hold.

$$\sup_{0 \le t < \infty} \max_{i,j} |x_i(t) - x_j(t)| < \infty, \qquad \lim_{t \to \infty} \max_{i,j} |v_i(t) - v_j(t)| = 0.$$

"To find sufficient conditions on parameters and initial data" leading to asymptotic flocking behavior for the C-S model:

$$\frac{dx_i}{dt} = v_i, \qquad \frac{dv_i}{dt} = \frac{K}{N} \sum_{j=1}^{N} \psi(|x_j - x_i|)(v_j - v_i),$$

subject to

$$(x_i, v_i)(0) = (x_{i0}, v_{i0}).$$

Global flocking theorem

- The Lyapunov functional approach: Assume that

$$\sum_{i=1}^{N} x_i(t) = 0, \qquad \sum_{i=1}^{N} v_i(t) = 0.$$

We set

$$||x|| := \Big(\sum_{i=1}^{N} ||x_i||^2\Big)^{\frac{1}{2}}, \qquad ||v|| := \Big(\sum_{i=1}^{N} ||v_i||^2\Big)^{\frac{1}{2}}.$$

Then $||x||$ and $||v||$ satisfy the SDDI:

$$\Big|\frac{d||x||}{dt}\Big| \le ||v||, \qquad \frac{d||v||}{dt} \le -\frac{K}{N}\psi(2||x||)||v||, \quad \text{a.e. } t \in \mathbb{R}$$

Introduce Lyapunov like functionals $\mathcal{E}_{\pm}$:

$$\mathcal{E}_{\pm}(||x||, ||v||) := ||v|| \pm \frac{K}{N} \int_0^{||x||} \psi(2s)ds.$$

- **Lemma**: Along the solution $(x(t), v(t))$ of the C-S model

$$(i)\ \mathcal{E}(||x(t)||, ||v(t)||) \leq \mathcal{E}(||x_0||, ||v_0||).$$
$$(ii)\ ||v(t)|| + \frac{K}{N}\Big| \int_{||x_0||}^{||x(t)||} \psi(2s)ds \Big| \leq ||v_0||.$$

- **Theorem**: (H-Liu '08)

If

$$||v_0|| < \frac{K}{2}\int_{||x_0||}^{\infty} \psi(2s)ds,$$

then $\exists\ x_M \geq 0$ such that

$$\sup_{t\geq 0}||x(t)|| \leq x_M, \qquad ||v(t)|| \leq ||v_0||e^{-\psi(x_M)t}.$$

where x_M satisfies the defining condition:

$$||v_0|| = \int_{||x_0||}^{x_M} \psi(2s)ds.$$

cf. 0. Cucker-Smale '07, H-Tadmor '08
1. N-dependence, general ψ.
2. Ahn-Choi-H-Lee '11: For N-free result, we set

$$||x||_\infty := \max_{1\leq i\leq N}||x_i||, \quad ||v||_\infty := \max_{1\leq i\leq N}||v_i||$$

3. This is a sufficient condition: What if the sufficient condition is violated ?

What's happening for some well-prepared setting ?

- **Definition** For a given initial configuration $\mathbf{z}^0 := (\mathbf{x}^0, \mathbf{v}^0)$, a non-negative constant $K_c = K_c(\mathbf{z}^0)$ is the critical coupling strength for global (mono-cluster) flocking if and only if the following two criteria hold.

1. If $K > K_c$, then the initial configuration $\mathbf{z}^0$ tends to a global flocking configuration asymptotically.
2. If $K \leq K_c$, then the initial configuration $\mathbf{z}^0$ does not tend to a global flocking configuration asymptotically.

Corollary of a global flocking theorem

- Case A: For a long range communication weight with $\int_0^\infty \psi(s)ds = \infty$,
$$K_c = 0.$$
- Case B: For a short range communication weight with $\int_0^\infty \psi(s)ds = \infty$,
$$K_c \leq \frac{2||v_0||}{\int_{||x_0||}^\infty \psi(2s)ds}.$$

Question: How small K_c should be ?

A two-particle system

Consider a two-particle system on the real line:

$$\dot{x}_1 = v_1, \quad \dot{x}_2 = v_2,$$
$$\dot{v}_1 = \frac{K}{2}\psi(|x_2 - x_1|)(v_2 - v_1), \quad \dot{v}_2 = \frac{K}{2}\psi(|x_1 - x_2|)(v_1 - v_2).$$

Then, the differences of $x =:= x_1 - x_2$ and $v := v_1 - v_2$ satisfy

$$\dot{x} = v, \quad \dot{v} = -K\psi(|x|)v, \quad \text{or} \quad dv = -K\psi(|x|)dx.$$

Integrating the above relation yields

$$v(t) = v_0 - K\int_{x_0}^{x(t)} \psi(|y|)dy.$$

- **Proposition**: Cho-H-Huang-Jin-Ko '15

For a given initial data (x_0, v_0), if the coupling strength K satisfies

$$K \leq \frac{v_0}{\int_{x_0}^{\infty} \psi(|y|)dy},$$

then there is no global flocking.

Proof. Suppose

$$v_0 \geq K \int_{x_0}^{\infty} \psi(|y|)dy.$$

Then, it follows from the previous relation that

$$\begin{aligned} v(t) &= v_0 - K \int_{x_0}^{x(t)} \psi(|y|)dy \geq K \int_{x_0}^{\infty} \psi(|y|)dy - K \int_{x_0}^{x(t)} \psi(|y|)dy \\ &= K \int_{x(t)}^{\infty} \psi(|y|)dy. \end{aligned}$$

Thus, we have

$$v(t) \geq K \int_{x(t)}^{\infty} \psi(|y|)dy.$$

Recall that the global flocking means

$$\sup_{t\geq 0} |x(t)| = x_\infty < \infty, \quad \lim_{t\to\infty} v(t) = v_\infty = 0.$$

Note that the above two conditions are not compatible, i.e., if $|x(t)| \leq x_\infty < \infty$, then

$$v(t) \geq K \int_{x(t)}^{\infty} \psi(|y|)dy \geq K \int_{x_\infty}^{\infty} \psi(|y|)dy,$$

which contradicts the fact that $\lim_{t\to\infty} v(t) = v_\infty = 0$. Therefore, there is no global flocking.

- **Corollary**: Let $\psi(s) = \frac{1}{(1+s)^\beta}$, $\beta > 1$ and (x_0, v_0) be a given initial data.

  1. If K satisfies

  $$K = \frac{v_0}{\int_{x_0}^{\infty} \psi(|y|)dy},$$

  then, we have

  $$\lim_{t\to\infty} |x(t)| = \infty, \quad \lim_{t\to\infty} |v(t)| = 0.$$

  2. If (x_0, v_0) satisfies

  $$K < \frac{v_0}{\int_{x_0}^{\infty} \psi(|y|)dy},$$

  then, we have

  $$\lim_{t\to\infty} |x(t)| = \infty, \quad \lim_{t\to\infty} |v(t)| =: v^\infty > 0.$$

For a two-oscillator system

$$K_c(x_0, v_0) := \frac{v_0}{\int_{x_0}^{\infty} \psi(|y|)dy}.$$

How about $N \geq 3$?

It might be very difficult to obtain exact critical coupling strength K_c for a many-body system. Thus, we will try to estimate a possible lower bound for K_c. Imagine what will happen as the coupling strength is varied from zero to a large value

Derivation of a positive lower bound for K_c

- Preparation A: Initial grouping

For a given non-flocking initial configuration $\mathcal{G} := \{(\mathbf{x}_i^0, \mathbf{v}_i^0)\}_{i=1}^N$:

$$\max_{i \neq j} ||\mathbf{v}_i^0 - \mathbf{v}_j^0|| > 0,$$

we partition initial configuration into subconfigurations $\mathcal{G}_1, \cdots, \mathcal{G}_n$ according to the initial data: for $\alpha = 1, \cdots, n$,

$$(\mathbf{x}_{\alpha i}, \mathbf{v}_{\alpha i}), (\mathbf{x}_{\alpha j}, \mathbf{v}_{\alpha j}) \in \mathcal{G}_\alpha \quad \Longleftrightarrow \quad v_{\alpha i}^0 = v_{\alpha j}^0, \qquad N_\alpha := |\mathcal{G}_\alpha|.$$

Membership is determined by their initial velocities

- Preparation B: Rearranging system

We rewrite the original system as

$$
\begin{aligned}
&\dot{\mathbf{x}}_{\alpha i}(t) = \mathbf{v}_{\alpha i}(t), \quad t > 0, \quad i = 1, 2, \cdots, N_\alpha,\\
&\dot{\mathbf{v}}_{\alpha i}(t) = \frac{K}{N}\sum_{k=1}^{N_\alpha} \psi(\|\mathbf{x}_{\alpha k}(t) - \mathbf{x}_{\alpha i}(t)\|)\big(\mathbf{v}_{\alpha k}(t) - \mathbf{v}_{\alpha i}(t)\big)\\
&\qquad\qquad + \frac{K}{N}\sum_{\beta\neq\alpha}\sum_{k=1}^{N_\beta} \psi(\|\mathbf{x}_{\beta k}(t) - \mathbf{x}_{\alpha i}(t)\|)\big(\mathbf{v}_{\beta k}(t) - \mathbf{v}_{\alpha i}(t)\big),\\
&(\mathbf{x}_{\alpha i}(0), \mathbf{v}_{\alpha i}(0)) = (\mathbf{x}^0_{\alpha i}, \mathbf{v}^0_{\alpha i}).
\end{aligned}
$$

- Preparation C: Dynamics of local averages and fluctuations

Introduce local averages and local fluctuations:

$$\mathbf{x}_\alpha^c := \frac{1}{N_\alpha}\sum_{i=1}^{N_\alpha}\mathbf{x}_{\alpha i}, \qquad \mathbf{v}_\alpha^c := \frac{1}{N_\alpha}\sum_{i=1}^{N_\alpha}\mathbf{v}_{\alpha i},$$
$$\hat{\mathbf{x}}_{\alpha i} := \mathbf{x}_{\alpha i} - \mathbf{x}_\alpha^c, \qquad \hat{\mathbf{v}}_{\alpha i} := \mathbf{v}_{\alpha i} - \mathbf{v}_\alpha^c.$$

$$\begin{cases} \dot{\mathbf{x}}^c_\alpha(t) = \mathbf{v}^c_\alpha(t), \quad t \geq 0, \quad \alpha = 1, 2, \cdots, n, \\ \dot{\mathbf{v}}^c_\alpha(t) = \dfrac{K}{NN_\alpha} \displaystyle\sum_{\beta \neq \alpha} \sum_{k=1}^{N_\beta} \sum_{i=1}^{N_\alpha} \psi(\|\mathbf{x}_{\beta k}(t) - \mathbf{x}_{\alpha i}(t)\|)\big(\mathbf{v}_{\beta k}(t) - \mathbf{v}_{\alpha i}(t)\big) \end{cases}$$

and

$$\begin{cases} \dot{\hat{\mathbf{x}}}_{\alpha i}(t) = \hat{\mathbf{v}}_{\alpha i}(t), \quad t \geq 0, \quad \alpha = 1, \cdots, n, \quad i = 1, 2, \cdots, N_\alpha, \\ \dot{\hat{\mathbf{v}}}_{\alpha i}(t) = -\dot{\mathbf{v}}^c_\alpha(t) + \dfrac{K}{N} \displaystyle\sum_{k=1}^{N_\alpha} \psi(\|\mathbf{x}_{\alpha k}(t) - \mathbf{x}_{\alpha i}(t)\|)\big(\hat{\mathbf{v}}_{\alpha k}(t) - \hat{\mathbf{v}}_{\alpha i}(t)\big) \\ \qquad\qquad + \dfrac{K}{N} \displaystyle\sum_{\beta \neq \alpha} \sum_{k=1}^{N_\beta} \psi(\|\mathbf{x}_{\beta k}(t) - \mathbf{x}_{\alpha i}(t)\|)\big(\mathbf{v}_{\beta k}(t) - \mathbf{v}_{\alpha i}(t)\big). \end{cases}$$

Our strategy for non-existence of a global flocking

For a given non-flocking initial configuration $(\mathbf{x}^0, \mathbf{v}^0)$,

- Initial stage: (From mixed configuration to segregated configuration) there exists $T_0 \geq 0$ such that

$$(\mathbf{x}_{\beta k}(T_0) - \mathbf{x}_{\alpha i}(T_0)) \cdot \left(\mathbf{v}^c_\beta(T_0) - \mathbf{v}^c_\alpha(T_0)\right) > 0.$$

In the absence of flocking coupling, i.e., $K = 0$, the free flow with $K = 0$ yields

$$\begin{aligned}
&(\mathbf{v}_{\alpha i}(t), \mathbf{v}_{\beta i}(t)) = (\mathbf{v}^c_\alpha(0), \mathbf{v}^c_\beta(0)) \quad \text{and} \\
&(\mathbf{x}_{\alpha i}(t), \mathbf{x}_{\beta i}(t)) = (\mathbf{x}^0_{\alpha i} + t\mathbf{v}^c_\alpha(0), \mathbf{x}^0_{\beta i} + t\mathbf{v}^c_\beta(0)), \quad t \geq 0.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&(\mathbf{x}_{\beta k}(t) - \mathbf{x}_{\alpha i}(t)) \cdot \left(\mathbf{v}^c_\beta(t) - \mathbf{v}^c_\alpha(t)\right) \\
&\qquad = (\mathbf{x}^0_{\beta i} - \mathbf{x}^0_{\alpha i}) \cdot (\mathbf{v}^c_\beta(0) - \mathbf{v}^c_\alpha(0)) + t|\mathbf{v}^c_\beta(0) - \mathbf{v}^c_\alpha(0)|^2.
\end{aligned}$$

- Intermediate stage: (From segregated configuration to close to non-mono-cluster flocking): there exists $T^* > T_0$ such that

$$\min_{\alpha\neq\beta,i,k}\left\{(\mathbf{v}_{\beta k}(t)-\mathbf{v}_{\alpha i}(t))\cdot e_{\beta\alpha}\right\} > \frac{\lambda_0}{2}, \text{for all } t\in[T_0,T^*),$$
$$\|\mathbf{x}_{\beta k}(t)-\mathbf{x}_{\alpha i}(t)\| \geq \frac{\lambda_0}{2}(t-T_0),$$

  where $e_{\beta\alpha}$ is the unit vector in the direction of $\mathbf{v}^c_\beta(T_0)-\mathbf{v}^c_\alpha(T_0)$.

- Final stage: (Emergence of non-mono-cluster configuration) By continuity argument, we show that

$$T^* = \infty$$

  and obtain the non-existence of mono-cluster flocking.

- **Theorem** H-Ko-Zhang-Zhang '16

Let $(\mathbf{x}(t), \mathbf{v}(t))$ be a global solution of the C-S model with short range communication weight ψ and initial data:

$$\max_{i \neq j} ||\mathbf{v}_i^0 - \mathbf{v}_j^0|| > 0.$$

Then, there exists K_0 such that if $K < K_0$,

$$\min_{\alpha \neq \beta, i, k} \sup_{t \geq 0} \|\mathbf{x}_{\alpha i}(t) - \mathbf{x}_{\beta k}(t)\| = \infty,$$
$$\min_{\alpha \neq \beta, i, k} \liminf_{t \to \infty} \|\mathbf{v}_{\alpha i}(t) - \mathbf{v}_{\beta k}(t)\| > 0.$$

In other words, mono-cluster flocking does not occur asymptotically. Moreover, all sub-ensembles are well separated.

For a given initial configuration $(\mathbf{x}^0, \mathbf{v}^0)$, we set

$$\lambda_0(\mathbf{x}^0, \mathbf{v}^0) := \frac{1}{2} \min_{\beta \neq \alpha} \|\mathbf{v}^c_\beta(0) - \mathbf{v}^c_\alpha(0)\|,$$
$$D(\mathbf{x}^0) := \max_{\beta \neq \alpha, i, k} \|\mathbf{x}^0_{\beta k} - \mathbf{x}^0_{\alpha i}\|,$$
$$T_0(\mathbf{x}^0, \mathbf{v}^0) := \max_{\beta \neq \alpha, i, k} \Big\{0, -\frac{\big(\mathbf{x}^0_{\beta k} - \mathbf{x}^0_{\alpha i}\big) \cdot \big(\mathbf{v}^c_\beta(0) - \mathbf{v}^c_\alpha(0)\big)}{\lambda_0^2}\Big\}.$$

For notational simplicity, we suppress the $(\mathbf{x}^0, \mathbf{v}^0)$ dependence of λ_0, T_0, K_0 as follows:

$$\lambda_0 := \lambda_0(\mathbf{x}^0, \mathbf{v}^0), \qquad T_0 := T_0(\mathbf{x}^0, \mathbf{v}^0).$$

Explicit representation of K_0

- If the initial configuration satisfies

$$\min_{\beta\neq\alpha,i,k}\big(\mathbf{x}_{\beta k}(0)-\mathbf{x}_{\alpha i}(0)\big)\cdot\big(\mathbf{v}^c_\beta(0)-\mathbf{v}^c_\alpha(0)\big)<0,$$

then, we set

$$K_0(\mathbf{x}^0,\mathbf{v}^0):=\min\Bigg\{\frac{\lambda_0^2}{12(1-\gamma_N)\sqrt{2M_2(0)}||\psi||_{L^1(\mathbb{R}_+)}},\\ \frac{\lambda_0}{16(1-\gamma_N)\sqrt{2M_2(0)}\psi(0)T_0},\\ \frac{\lambda_0^2}{(1-\gamma_N)\sqrt{2M_2(0)}\psi(0)\big(D(\mathbf{x}^0)+\sqrt{2M_2(0)}T_0\big)}\Bigg\},$$

where we have used the simplified notation $\gamma_N:=\frac{\min N_\beta}{N}$.

- If the initial configuration satisfies

$$\min_{\beta\neq\alpha,i,k}\left(\mathbf{x}_{\beta k}(0)-\mathbf{x}_{\alpha i}(0)\right)\cdot\left(\mathbf{v}^c_{\beta}(0)-\mathbf{v}^c_{\alpha}(0)\right)\geq 0, \qquad (3)$$

then, we set

$$K_0(\mathbf{x}^0,\mathbf{v}^0):=\frac{\lambda_0^2}{6(1-\gamma_N)\sqrt{2M_2(0)}||\psi||_{L^1(\mathbb{R}_+)}}.$$

From infinite variance to zero variance in a random environment

This is a joint work with J. Jeong, S. Noh and Q. Xiao

First try with additive noises

Recall the Cucker-Smale model:

$$\dot{x}_i = v_i, \quad \dot{v}_i = \frac{K}{N}\sum_{j=1}^{N}\psi_{ij}(x)(v_j - v_i).$$

Consider a noise perturbation of the C-S model: H-Lee-Levy '09:

$$\begin{aligned} dx_t^i &= v_t^i dt, \\ dv_t^i &= \Big[\frac{K}{N}\sum_{j=1}^{N}\psi_{ij}(v_t^j - v_t^i)\Big]dt + \frac{1}{N}\sum_{j=1}^{N}\sigma_{ij}dW_t^{ij}. \end{aligned}$$

Consider a simple situation:

$$\sigma_{ij} = \sqrt{2\sigma}, \quad dW_t^{ij} = dW_t.$$

Then under this simplified situation, the system becomes

$$\begin{aligned} dx_t^i &= v_t^i dt, \\ dv_t^i &= \Big[\frac{K}{N}\sum_{j=1}^N \psi_{ij}(v_t^j - v_t^i)\Big] dt + \sqrt{2\sigma} dW_t. \end{aligned}$$

- The CSFP: Bolley-Canizo-Carrillo '11

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (L[f]f) = \sigma \Delta_v f.$$

Does this system exhibit flocking ?

For $\psi = 1$ and suitable normalization conditions

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (Kvf + \sigma \nabla_v f).$$

Then, it is easy to check that the above equation has a space-homogeneous equilibrium f_∞:

$$f_\infty(v) = e^{-\frac{K}{2\sigma}|v|^2}, \quad v \in \mathbb{R}^d;$$

therefore, there is no emergent velocity alignment for any positive $K > 0$.

Second try with multiplicative noises

Recall the Cucker-Smale model:

$$\dot{x}_i = v_i, \quad \dot{v}_i = \frac{K}{N}\sum_{j=1}^{N} \bar{\psi}_{ij}(x)(v_j - v_i).$$

Consider a noise perturbation of the communication weight $\bar{\psi}$:

$$\bar{\psi}_{ij} = \psi_{ij} + \text{Noises} = \psi_{ij} + \frac{\sigma_{ij}}{K}\dot{W}_{ij}.$$

Then the C-S model becomes

$$\begin{aligned} dx_t^i &= v_t^i dt, \\ dv_t^i &= \Big[\frac{K}{N}\sum_{j=1}^{N}\psi_{ij}(v_t^j - v_t^i)\Big]dt + \frac{1}{N}\sum_{j=1}^{N}\sigma_{ij}(v_t^j - v_t^i)dW_t^{ij}. \end{aligned}$$

Consider an extreme situation:

$$\sigma_{ij} = \sigma, \quad dW_t^{ij} = dW_t.$$

Then under this simplified situation, the system becomes

$$\begin{aligned} dx_t^i &= v_t^i dt, \\ dv_t^i &= \Big[\frac{K}{N}\sum_{j=1}^{N}\psi_{ij}(v_t^j - v_t^i)\Big]dt + \frac{\sigma}{N}\sum_{j=1}^{N}(v_t^j - v_t^i)dW_t. \end{aligned}$$

Since the total momentum is conserved,

$$d\Big(\sum_{i=1}^{N} v^i\Big) = 0,$$

we may assume

$$\sum_{i=1}^{N} v^i = 0, \quad \sum_{i=1}^{N} x^i = 0.$$

- A stochastic C-S model

$$\begin{aligned} dx_t^i &= v_t^i dt, \\ dv_t^i &= \Big[\frac{K}{N}\sum_{j=1}^{N}\psi_{ij}(v_t^j - v_t^i)\Big]dt - \sigma v_t^i dW_t. \end{aligned}$$

We set

$$\mathcal{X}_t := \sum_{i=1}^{N} |x_t^i|^2, \quad \mathcal{V}_t := \sum_{i=1}^{N} |v_t^i|^2.$$

Then variance processes $(\mathcal{X}_t, \mathcal{V}_t)$ satisfy

$$\begin{aligned} d\mathcal{X}_t &\leq 2\sqrt{\mathcal{X}_t}\sqrt{\mathcal{V}_t}dt, \\ d\mathcal{V}_t &\leq \Big(-2K\psi(2\mathcal{X}_t) + \sigma^2 \Big)\mathcal{V}_t dt + 2\sigma\mathcal{V}_t dW_t. \end{aligned}$$

- **Theorem**: Ahn-Ha '10

Suppose that ψ satisfies

$$\min_{s\geq 0} \psi(s) \geq \psi_* > 0.$$

Then $\mathcal{V}_t$ satisfies

$$\mathcal{V}_t \leq \mathcal{V}_0 \exp\Big(-(2K\psi_* + \sigma^2)t + 2\sigma W_t \Big), \quad \text{a.e.}$$

The Cucker-Smale-Fokker-Planck equation

- Cauchy problem for the kinetic CS-FP equation:

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (L[f]f) = \sigma \Delta_v (|v - v^c|^2 f), \quad x, v \in^d, \quad t > 0,$$
$$L[f](x, v, t) := -K \int_{\mathbb{R}^{2d}} \psi(|x - y|)(v - v_*) f(y, v_*, t) dv_* dy,$$
$$f(x, v, 0) = f_0(x, v),$$

where v^c is the average mean velocity:

$$v^c(t) := \frac{\int_{2d} vfdvdx}{\int_d fdvdx}.$$

- Lyapunov functional:

$$\mathcal{L}(f(t)) := \int_{2d} |v - v^c(t)|^2 f dv dx = \int_{2d} |v - v^c(0)|^2 f dv dx,$$

- Zero convergence of $\mathcal{L}(f(t))$: For any $\varepsilon > 0$, we have

$$\begin{aligned} \mathcal{L}(f(t)) &= \int_{2d} |v - v^c(t)|^2 f dv dx \geq \int_{|v - v^c(0)| > \varepsilon} |v - v^c(0)|^2 f dv dx \\ &\geq \varepsilon^2 \int_{|v - v^c(0)| > \varepsilon} f dv dx = \varepsilon^2 \mathbb{P}[|v - v^c(0)| > \varepsilon]. \end{aligned}$$

This implies

$$\lim_{t \to \infty} \mathbb{P}[|v - v^c(0)| > \varepsilon] \leq \frac{1}{\varepsilon^2} \lim_{t \to \infty} \mathcal{L}(f(t)) = 0.$$

- Flocking estimate:

$$
\begin{aligned}
&\frac{d}{dt}\int_{2d}|v-v^c|^2 f dv dx \\
&\quad = \underbrace{-K\int_{4d}\psi(|x-y|)|v-v_*|^2 f(y,v_*,t)f(x,v,t)dv_* dv dy dx}_{\text{effect of the velocity alignment forcing}} \\
&\quad + \underbrace{2d\sigma\int_{2d}|v-v^c|^2 f dv dx}_{\text{effect of the nonuniform diffusion}} .
\end{aligned}
$$

- **Theorem**: H-Jeong-Noh-Xiao '16 Suppose that

$$\psi_m \leq \psi(s) \leq \psi_M, \quad s \geq 0, \quad \int_{2d} (1+|v|^2) f(x,v,t)\,dvdx < \infty, \quad t \geq 0.$$

Then, the following estimates hold.

1. If $K > \dfrac{d\sigma}{\psi_m \|f_0\|_{L^1}}$, there exists a positive constant $\lambda_m := 2\left(K\psi_m\|f_0\|_{L^1} - d\sigma\right)$ such that

$$\mathcal{L}(f(t)) \leq \mathcal{L}(f_0) e^{-\lambda_m t}, \quad t \geq 0.$$

2. If $K < \dfrac{d\sigma}{\psi_M \|f_0\|_{L^1}}$, there exists a positive constant $\lambda_M := 2\left(d\sigma - K\psi_M\|f_0\|_{L^1}\right)$ such that

$$\mathcal{L}(f(t)) \geq \mathcal{L}(f_0) e^{\lambda_M t}, \quad t \geq 0.$$

- **Remark**: For $\psi \equiv 1$,

$$\lim_{t\to\infty} \mathcal{L}(f(t)) = \begin{cases} \infty, \quad K < K_c, & \text{subcritical regime,} \\ \mathcal{L}(f_0), \quad K = K_c, & \text{critical regime,} \\ 0, \quad K > K_c & \text{supercritical regime,} \end{cases}$$

where $K_c := \frac{d\sigma}{\|f_0\|_{L^1}}$

Epilogue

- I have provided two critical phenomena arising from the dynamics of the Cucker-Smale model:

  Global flocking v.s. local flocking,
  flocking coupling v.s. multiplicative noises

- Possible future directions: Extension of flocking theory to the active particle systems (e.g., Thermodynamically consistent C-S model)

To Prof. Aoki and Sone, happy birthday !!!