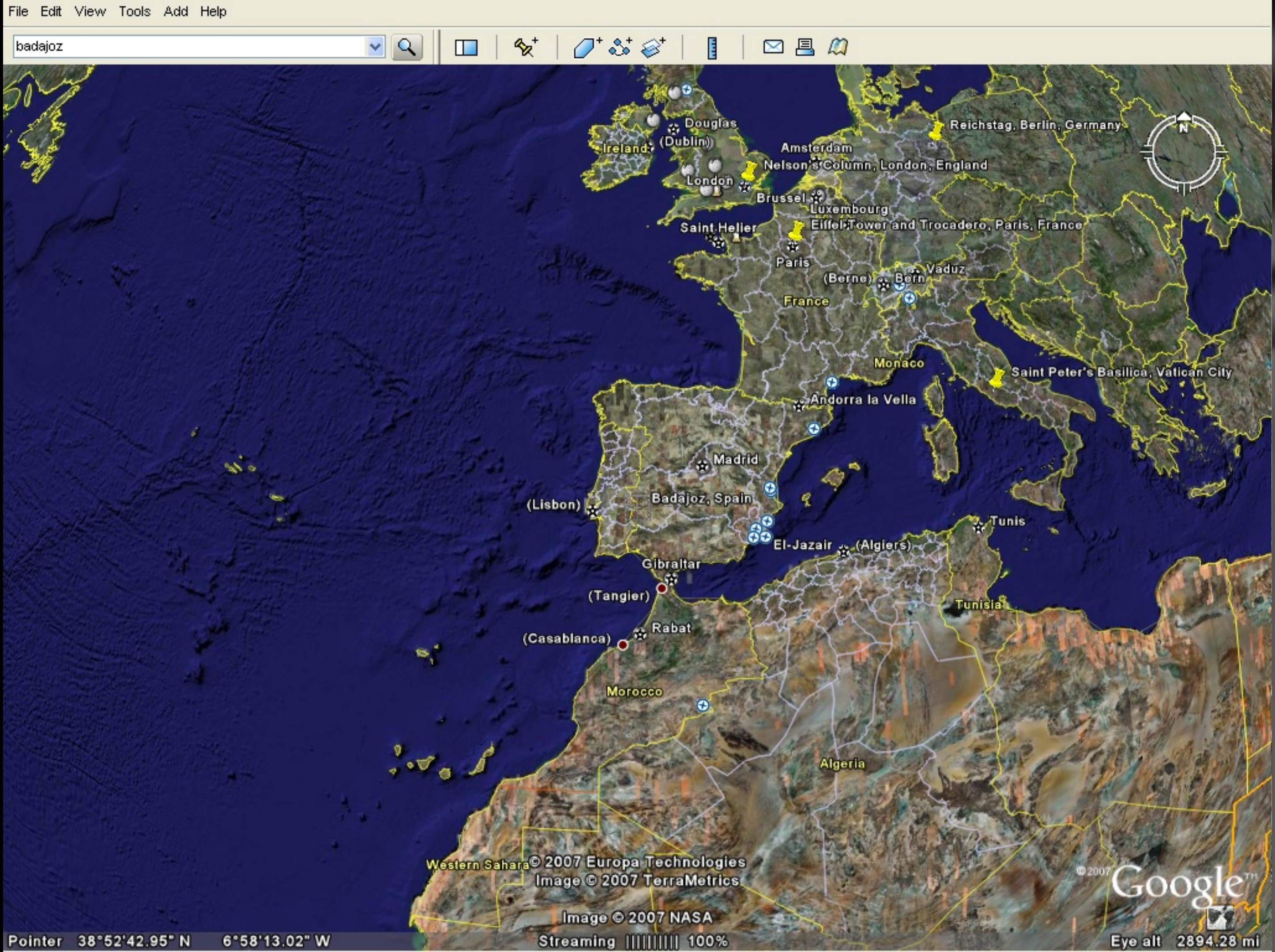
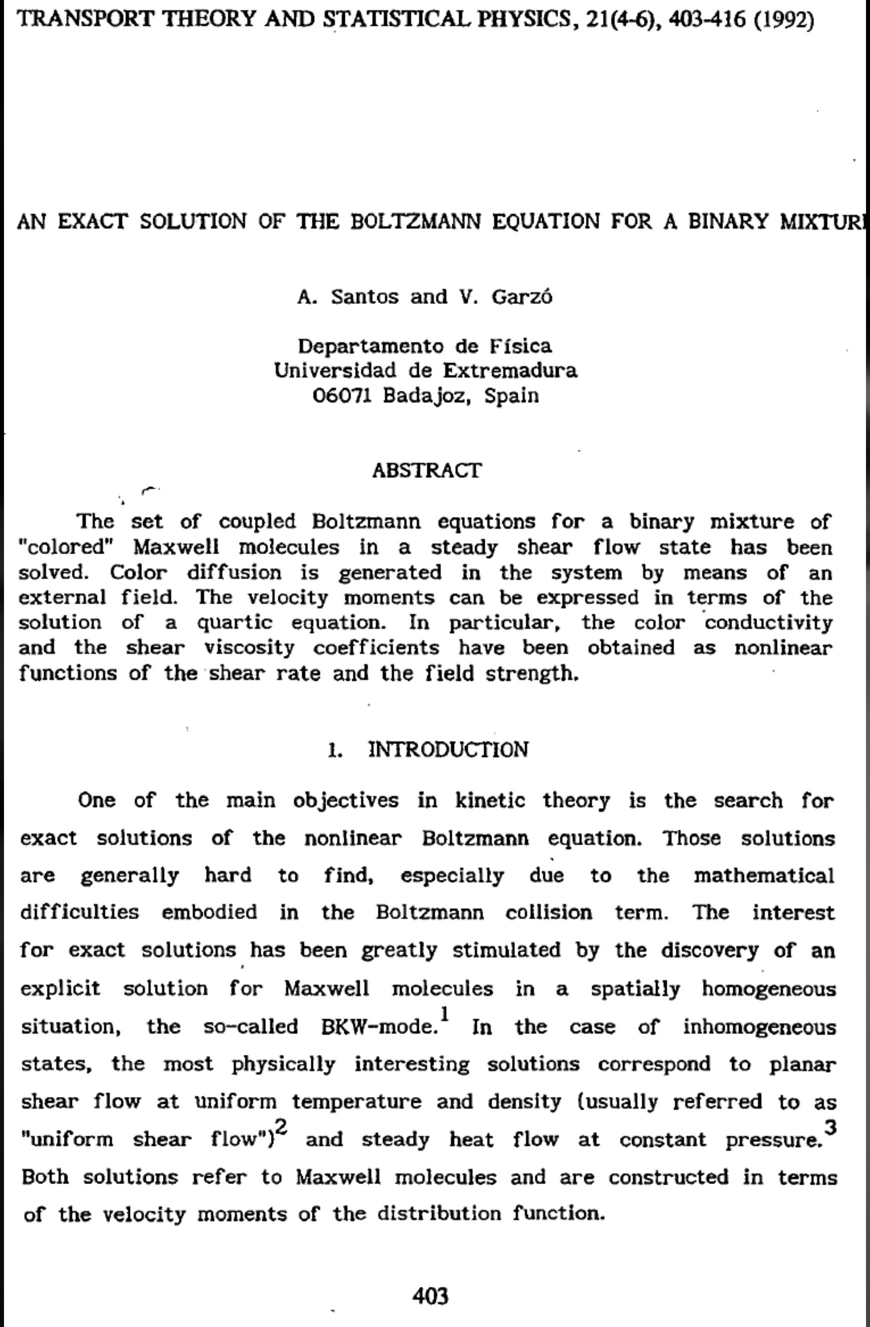
Hydrodynamics for granular flow at low density:
Navier-Stokes and Burnett constitutive equations

Andrés Santos
Universidad de Extremadura. Badajoz, Spain

My Badajoz-Kyoto connection: The last 25 years



October 1991: Fourth International Workshop on Mathematical Aspects of Fluid and Plasma Dynamics, Kyoto



TRANSPORT THEORY AND STATISTICAL PHYSICS, 21(4-6), 403-416 (1992)

AN EXACT SOLUTION OF THE BOLTZMANN EQUATION FOR A BINARY MIXTUR

A. Santos and V. Garzó

Departamento de Física
Universidad de Extremadura
06071 Badajoz, Spain

ABSTRACT

The set of coupled Boltzmann equations for a binary mixture of "colored" Maxwell molecules in a steady shear flow state has been solved. Color diffusion is generated in the system by means of an external field. The velocity moments can be expressed in terms of the solution of a quartic equation. In particular, the color conductivity and the shear viscosity coefficients have been obtained as nonlinear functions of the shear rate and the field strength.

1. INTRODUCTION

One of the main objectives in kinetic theory is the search for exact solutions of the nonlinear Boltzmann equation. Those solutions are generally hard to find, especially due to the mathematical difficulties embodied in the Boltzmann collision term. The interest for exact solutions has been greatly stimulated by the discovery of an explicit solution for Maxwell molecules in a spatially homogeneous situation, the so-called BKW-mode.[1] In the case of inhomogeneous states, the most physically interesting solutions correspond to planar shear flow at uniform temperature and density (usually referred to as "uniform shear flow")[2] and steady heat flow at constant pressure.[3] Both solutions refer to Maxwell molecules and are constructed in terms of the velocity moments of the distribution function.

403

November 1997:
Prof. Sone visits Badajoz

From: Yoshio Sone <sone@sum
To: "'Andres Santos'" <andres@u
Subject: RE: Some small details
Date: Wed, 12 Nov 1997 09:44:3

Dear Professor Santos,
Thank you very much for your ac
I am very happy to give a semina
The title is
" Flows induced by temperature f
and its finite effect on the behav

The video of flows induced by ter
Wesdnesday is convenient to me
Best Regards,
Yoshio Sone

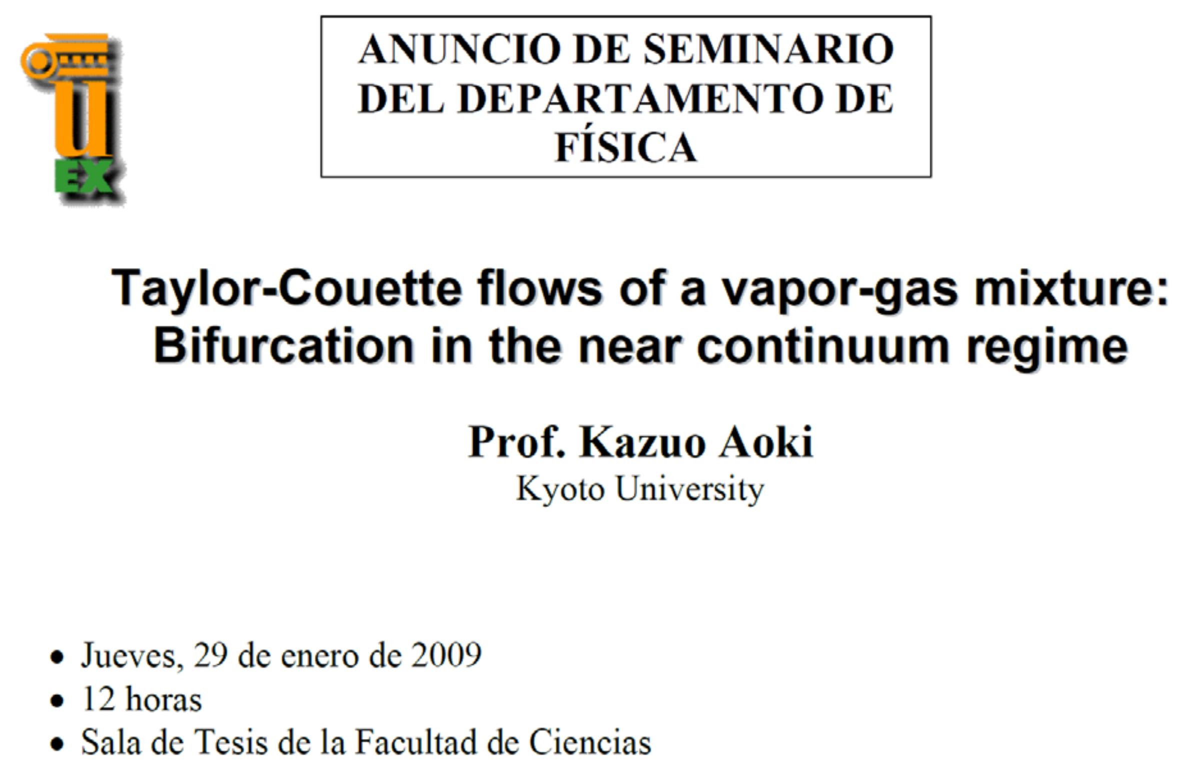
◆ Demonstration of a rarefied gas flow induced near the edge of a uniformly heated plate

Index
- ·Title
- ·The experimental apparatus
- ·The plate with a heater
- ·The vanes detecting a horizontal wind
- ·The vanes detecting a vertical wind
- ·Experiment I :Horizontal flow induced near the vertical edge of the plate
    The plate set vertically with its longer edges in the vertical direction
    The general view
    Atmospheric pressure[two windmills]
    ４００Ｐａ[two windmills]
    １００Ｐａ[two windmills]
    ６０Ｐａ[two windmills]
    ４０Ｐａ[two windmills]
    ２０Ｐａ[two windmills]
    １０Ｐａ[two windmills]
    ５Ｐａ[two windmills]
    １０Ｐａ(right side/left side)[with same windmill]
- ·Experiment II :Vertical flow induced near the horizontal edge of the plate
    The plate set vertically with its longer edges in the horizontal direction
    The general view
    Atmospheric pressure
    ６０Ｐａ
    ４０Ｐａ
    ２０Ｐａ
    １０Ｐａ

March-April 1999: JSPS Fellow, Kyoto

July 2008:
RGD 26, Kyoto

January 2009:
Prof. Aoki visits Badajoz



ANUNCIO DE SEMINARIO DEL DEPARTAMENTO DE FÍSICA

Taylor-Couette flows of a vapor-gas mixture: Bifurcation in the near continuum regime

Prof. Kazuo Aoki
Kyoto University

- Jueves, 29 de enero de 2009
- 12 horas
- Sala de Tesis de la Facultad de Ciencias

July 2009:
YITP long-term workshop, Kyoto

May 2016:
Sone & Aoki's *fest*

Hydrodynamics for granular flow at low density: **Navier-Stokes and Burnett constitutive equations**

Andrés Santos
Universidad de Extremadura. Badajoz, Spain

Outline

- Introduction. Granular hydrodynamics
- The inelastic rough hard-sphere model (IRHSM). Navier-Stokes coefficients
- The inelastic Maxwell model (IMM) Burnett coefficients
- Conclusions

Outline

- Introduction. Granular hydrodynamics
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What is a granular material?

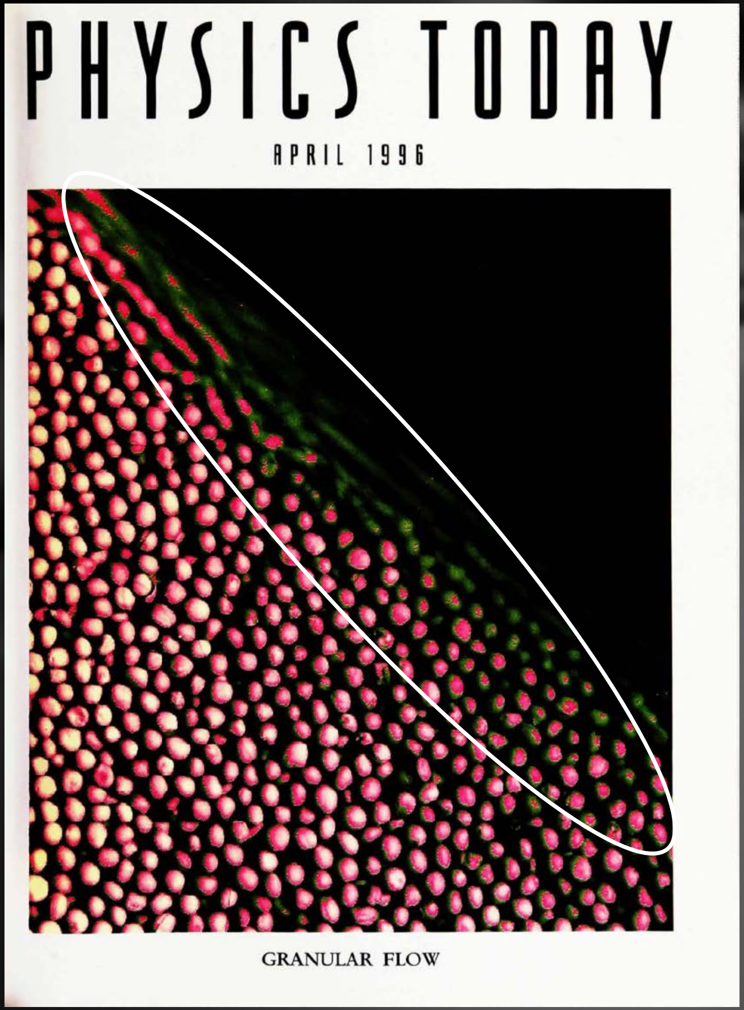
- It is a conglomeration of discrete solid, macroscopic particles characterized by a loss of energy whenever the grains collide.
- The constituents must be large enough such that they are not subject to thermal motion fluctuations. Thus, the lower size limit for grains is about 1 μm.

What is a granular *fluid*?

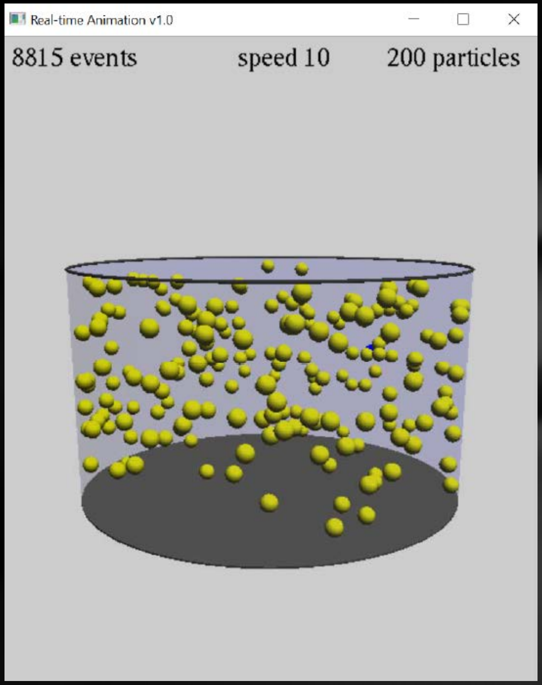
When the granular matter is driven and energy is fed into the system (e.g., by shaking) such that the grains are not in constant contact with each other, the granular material is said to *fluidize*.

Granular gas: Mean free path much larger than the grain size

KINETIC THEORY



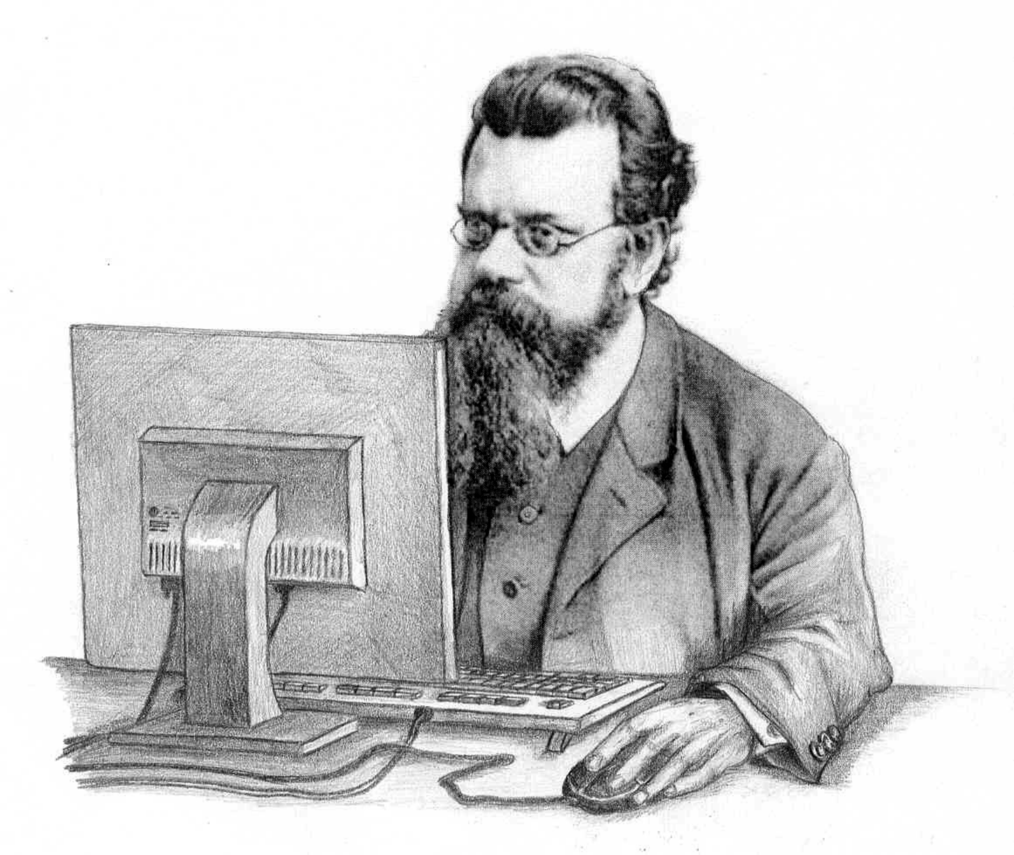
Granular gas: Dissipative collisions



8815 events speed 10 200 particles

Demo by Sergei Mechov

KINETIC THEORY



(Cartoon by Bernhard Reischl, University of Vienna)

Boltzmann equation:

$$\partial_t f(\mathbf{r},\mathbf{v},t) + \mathbf{v}\cdot\nabla f(\mathbf{r},\mathbf{v},t) = \underbrace{J[\mathbf{r},\mathbf{v},t|f]}_{\text{Dissipative collisions}}$$

HYDRODYNAMICS

Hydrodynamic balance equations

$$\left.\begin{array}{lrl} \text{Mass conservation:} & \mathcal{D}_t n + n\nabla\cdot\mathbf{u} = & 0 \\ \text{Momentum conservation:} & \rho\mathcal{D}_t\mathbf{u} + \nabla\cdot\mathsf{P} = & 0 \\ \text{Energy } \textit{dissipation}\text{:} & \mathcal{D}_t T + \dfrac{2}{n\underbrace{d}_{\text{Dimensionality}}}(\nabla\cdot\mathbf{q} + \mathsf{P}:\nabla\mathbf{u}) = & -\underbrace{\zeta}_{\text{Cooling rate}} T \end{array}\right\}$$

$$(\mathcal{D}_t \equiv \partial_t + \mathbf{u}\cdot\nabla)$$

Navier-Stokes (NS) constitutive equations

Claude-Louis Navier
(1785-1836)

George Gabriel Stokes
(1819-1903)

Jean-Baptiste Joseph Fourier
(1768-1830)

$$P_{ij} = p\delta_{ij} - \eta\left(\nabla_i u_j + \nabla_j u_i - \frac{2}{d}\delta_{ij}\nabla\cdot\mathbf{u}\right) - \eta_b\delta_{ij}\nabla\cdot\mathbf{u}$$

Shear viscosity

Bulk viscosity

Dufour-like coefficient

$$\mathbf{q} = -\lambda\nabla T - \mu\nabla n$$

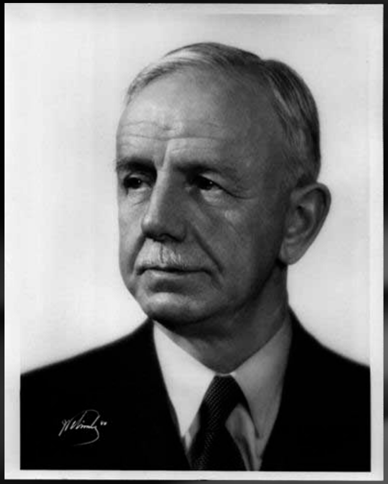
Thermal conductivity

$$\zeta = \zeta^{(0)} - \xi\nabla\cdot\mathbf{u}$$

Cooling rate transport coefficient

Methodology: Chapman-Enskog method

Sydney Chapman
(1888-1970)

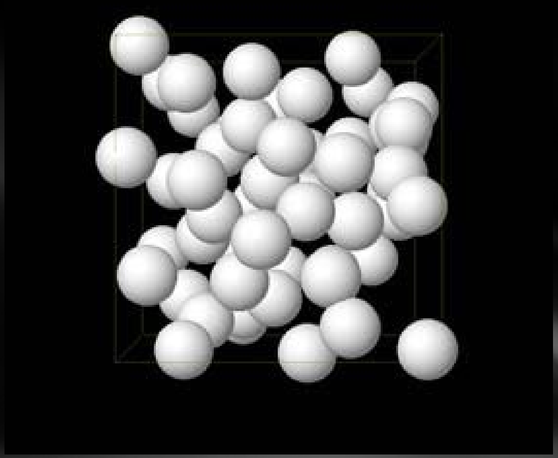


David Enskog
(1884-1947)

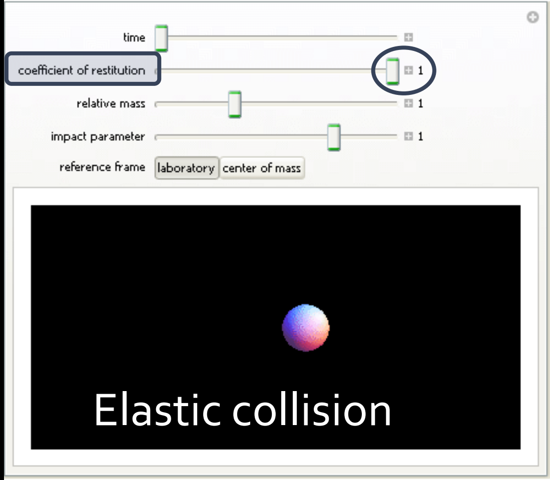
$$\boxed{f = f^{(0)} + \underbrace{\epsilon f^{(1)}}_{\text{Navier-Stokes}} + \underbrace{\epsilon^2 f^{(2)}}_{\text{Burnett}} + \cdots}, \quad \epsilon \sim \nabla$$

MODELS OF GRAINS

Standard model of a granular gas: A gas of identical *inelastic smooth* hard spheres (ISHSM)



Constant coefficient of *normal* restitution α



time

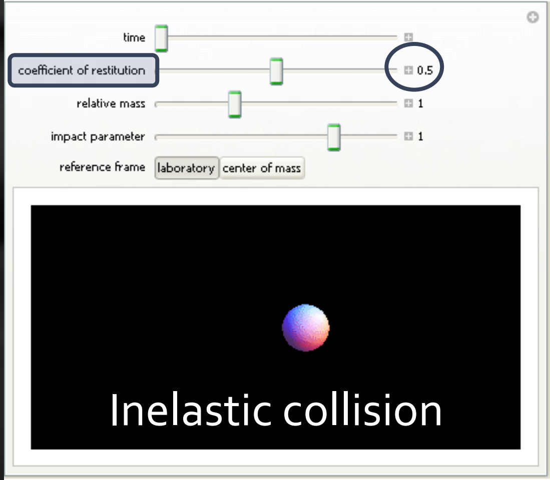
coefficient of restitution 1

relative mass 1

impact parameter 1

reference frame

Elastic collision



time

coefficient of restitution 0.5

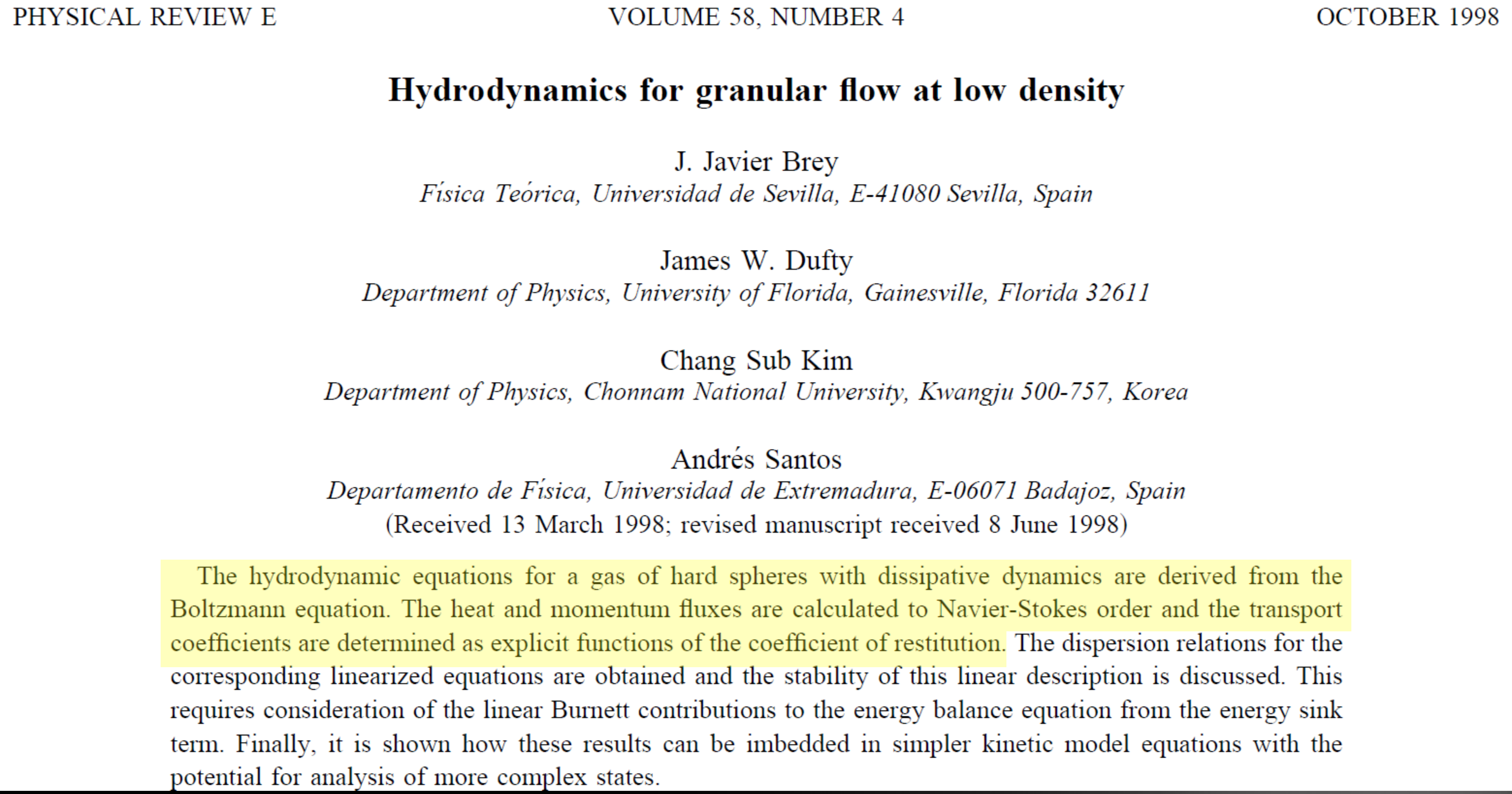
relative mass 1

impact parameter 1

reference frame

Inelastic collision

http://demonstrations.wolfram.com/InelasticCollisionsOfTwoSpheres/



PHYSICAL REVIEW E VOLUME 58, NUMBER 4 OCTOBER 1998

Hydrodynamics for granular flow at low density

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(Received 13 March 1998; revised manuscript received 8 June 1998)

The hydrodynamic equations for a gas of hard spheres with dissipative dynamics are derived from the Boltzmann equation. The heat and momentum fluxes are calculated to Navier-Stokes order and the transport coefficients are determined as explicit functions of the coefficient of restitution. The dispersion relations for the corresponding linearized equations are obtained and the stability of this linear description is discussed. This requires consideration of the linear Burnett contributions to the energy balance equation from the energy sink term. Finally, it is shown how these results can be imbedded in simpler kinetic model equations with the potential for analysis of more complex states.

Hydrodynamic order

Burnett	**2014 (*d*D, Exact)**	**?**	**?**
Navier-Stokes (NS)	2003 (*d*D, Exact)	1998 (Sonine appr.)	**2014 (3D, Sonine appr.)**
	Inelastic Maxwell model (IMM)	Inelastic smooth hard-sphere model (ISHSM)	Inelastic rough hard-sphere model (IRHSM)

Model complexity

Outline

- Introduction. Granular hydrodynamics
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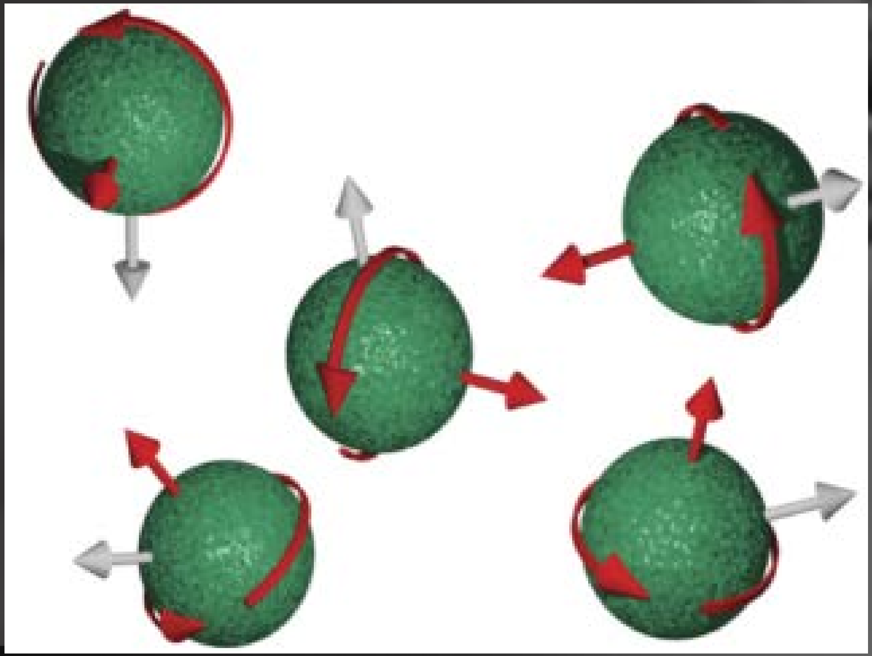
- Conclusions

Inelastic rough hard-sphere model (IRHSM)

Material parameters:

- Mass m
- Diameter σ
- Moment of inertia I ($\kappa=4I/m\sigma^2$)
- Coefficient of normal restitution α
- Coefficient of tangential restitution β
- α =1 for perfectly elastic particles
- β=-1 for perfectly smooth particles
- β=+1 for perfectly rough particles



This model unveils the inherent breakdown of equilibrium and energy equipartition in granular fluids, even in *homogeneous* and isotropic states

Collision rules

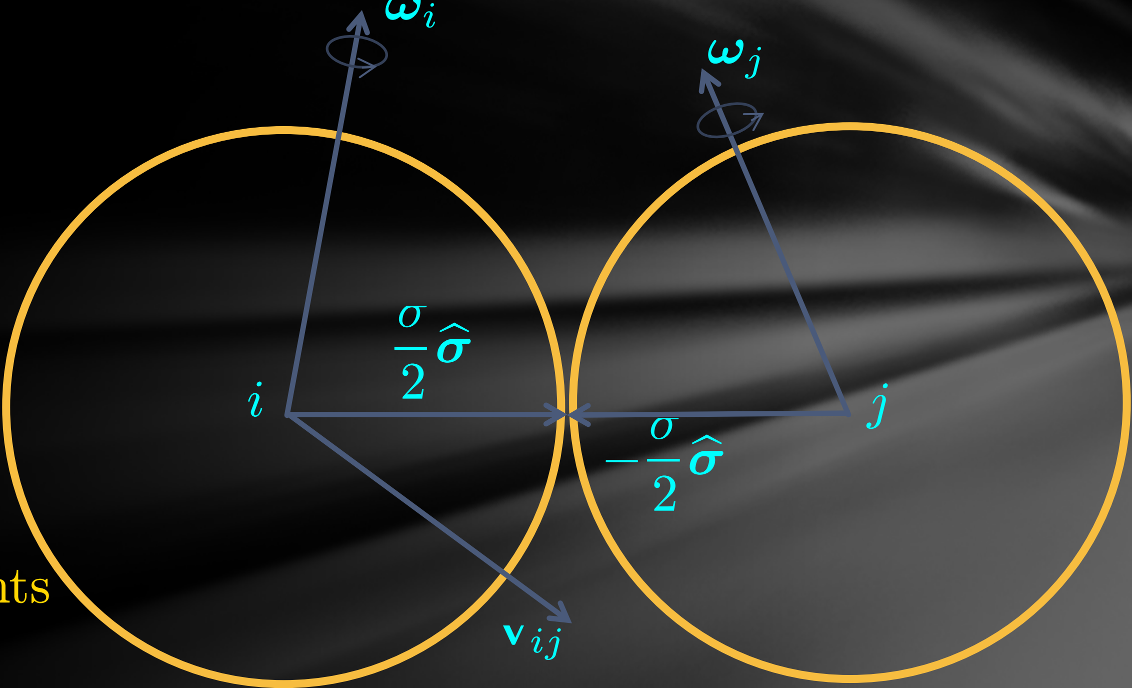
Cons. linear momentum:

$$\mathbf{v}'_i + \mathbf{v}'_j = \mathbf{v}_i + \mathbf{v}_j$$

Cons. angular momentum:

$$I\boldsymbol{\omega}'_{i,j} \mp m\frac{\sigma_i}{2}\widehat{\boldsymbol{\sigma}} \times \mathbf{v}'_{i,j}$$

$$= I\boldsymbol{\omega}_{i,j} \mp m\frac{\sigma_i}{2}\widehat{\boldsymbol{\sigma}} \times \mathbf{v}_{i,j}$$



Relative velocity of the points of the spheres at contact:

$$\overline{\mathbf{v}}_{ij} = \mathbf{v}_{ij} - \frac{\sigma}{2}\widehat{\boldsymbol{\sigma}} \times (\boldsymbol{\omega}_i + \boldsymbol{\omega}_j)$$

$$\boxed{\widehat{\boldsymbol{\sigma}} \cdot \overline{\mathbf{v}}'_{ij} = -\alpha\widehat{\boldsymbol{\sigma}} \cdot \overline{\mathbf{v}}_{ij}, \quad \widehat{\boldsymbol{\sigma}} \times \overline{\mathbf{v}}'_{ij} = -\beta\widehat{\boldsymbol{\sigma}} \times \overline{\mathbf{v}}_{ij}}$$

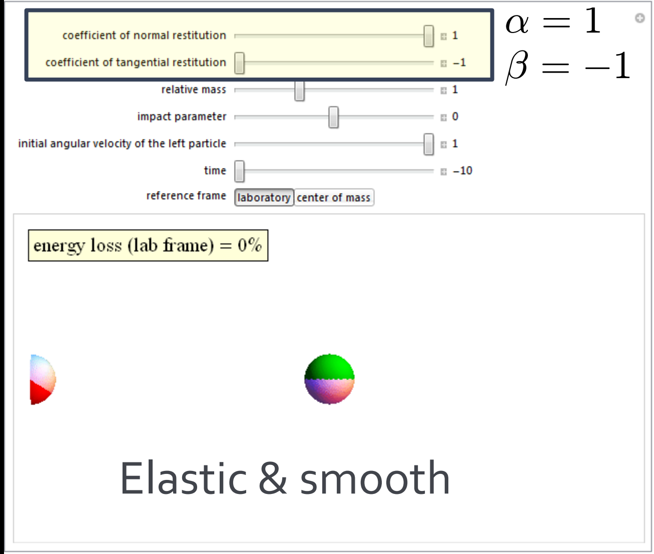
Energy collisional loss

$$E_{ij} = \frac{1}{2}mv_i^2 + \frac{1}{2}mv_j^2 + \frac{1}{2}I\omega_i^2 + \frac{1}{2}I\omega_j^2$$

$$\begin{aligned} E'_{ij} - E_{ij} &= -(1-\alpha^2) \times \cdots \\ &\quad -(1-\beta^2) \times \cdots \end{aligned}$$

Energy is conserved *only* if the spheres are

- elastic (α=1) and
- either
    - perfectly smooth (β=-1) or
    - perfectly rough (β=+1)



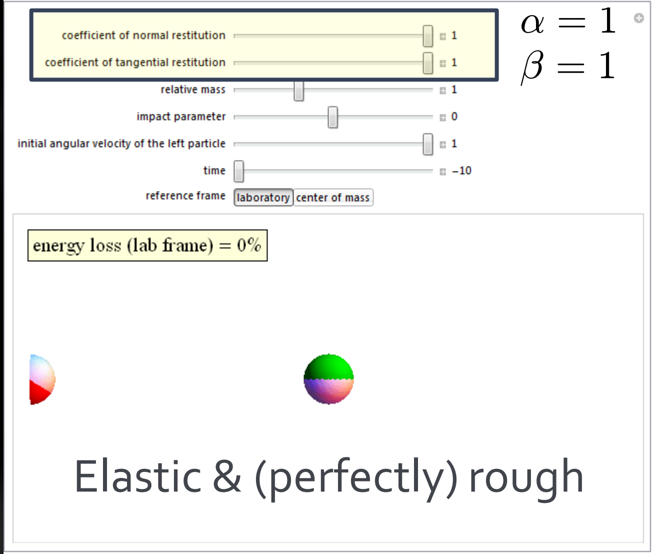
$\alpha = 1$

$\beta = -1$

coefficient of normal restitution 1
coefficient of tangential restitution −1
relative mass 1
impact parameter 0
initial angular velocity of the left particle 1
time −10
reference frame [✓ laboratory] center of mass

energy loss (lab frame) = 0%

Elastic & smooth



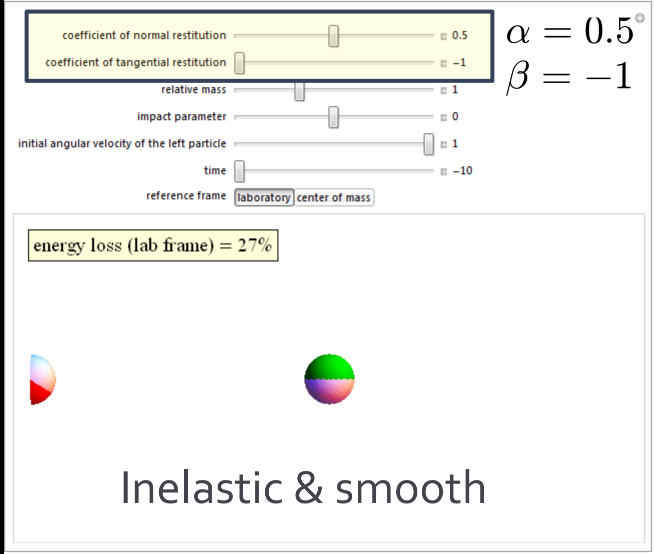
$\alpha = 1$

$\beta = 1$

coefficient of normal restitution 1
coefficient of tangential restitution 1
relative mass 1
impact parameter 0
initial angular velocity of the left particle 1
time −10
reference frame [✓ laboratory] center of mass

energy loss (lab frame) = 0%

Elastic & (perfectly) rough



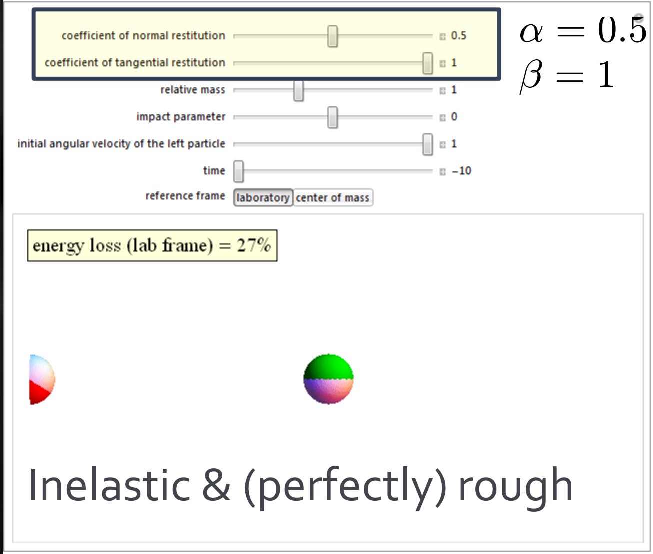
$\alpha = 0.5$

$\beta = -1$

coefficient of normal restitution 0.5
coefficient of tangential restitution −1
relative mass 1
impact parameter 0
initial angular velocity of the left particle 1
time −10
reference frame [✓ laboratory] center of mass

energy loss (lab frame) = 27%

Inelastic & smooth



$\alpha = 0.5$

$\beta = 1$

coefficient of normal restitution 0.5
coefficient of tangential restitution 1
relative mass 1
impact parameter 0
initial angular velocity of the left particle 1
time −10
reference frame [✓ laboratory] center of mass

energy loss (lab frame) = 27%

Inelastic & (perfectly) rough

http://demonstrations.wolfram.com/InelasticCollisionsOfTwoRoughSpheres/

G. M. Kremer, A. S., and V. Garzó, Phys. Rev. E **90**, 022205 (2014)

Explicit expressions

$\widetilde{\alpha}=\frac{1+\alpha}{2},\ \widetilde{\beta}=\frac{1+\beta}{2}\frac{\kappa}{\kappa+1}$

HCS quantities

$T_t^{(0)}/T=\tau_t=\frac{2}{1+\theta},\ T_r^{(0)}/T=\tau_r=\frac{2\theta}{1+\theta}$

$\theta=\sqrt{1+h^2}+h,\ h\equiv\frac{(1+\kappa)^2}{2\kappa(1+\beta)^2}\left[1-\alpha^2-(1-\beta^2)\frac{1-\kappa}{1+\kappa}\right]$

$\nu=\frac{16}{5}\sigma^2 n\sqrt{\pi\tau_t T/m}$

$\zeta^{(0)}/\nu=\zeta^*=\frac{5}{12}\frac{1}{1+\theta}\left[1-\alpha^2+(1-\beta^2)\frac{\theta+\kappa}{1+\kappa}\right]$

NS coefficients

$\eta=(n\tau_t T/\nu)/(\nu_\eta^*-\frac{1}{2}\zeta^*)$

$\eta_b=(n\tau_t\tau_r T/\nu)\gamma_E$

$\lambda=\tau_t\lambda_t+\tau_r\lambda_r,\ \lambda_t=\frac{5}{2}(n\tau_t T/m\nu)\gamma_{A_t},\ \lambda_r=\frac{3}{2}(n\tau_t T/m\nu)\gamma_{A_r}$

$\mu=\mu_t+\mu_r,\ \mu_t=\frac{5}{2}(\tau_t^2 T^2/m\nu)\gamma_{B_t},\ \mu_r=\frac{3}{2}(\tau_t\tau_r T^2/m\nu)\gamma_{B_r}$

$\xi=\frac{1}{2}(\tau_t\xi_t+\tau_r\xi_r)=\gamma_E\Xi,\ \xi_t=\gamma_E\Xi_t,\ \xi_r=\gamma_E\Xi_r$

Auxiliary quantities

$\nu_\eta^*=(\widetilde{\alpha}+\widetilde{\beta})(2-\widetilde{\alpha}-\widetilde{\beta})+\widetilde{\beta}^2\theta/6\kappa$

$\gamma_E=\frac{2}{3}(\Xi_t-\Xi_r-\zeta^*)^{-1}$

$\Xi_t=\frac{5}{8}\tau_r\left[1-\alpha^2+(1-\beta^2)\frac{\kappa}{1+\kappa}-\frac{\kappa}{3}(\theta-5)\left(\frac{1+\beta}{1+\kappa}\right)^2\right]$

$\Xi_r=\frac{5}{8}\tau_t\frac{1+\beta}{1+\kappa}\left[\frac{\theta-2}{3}(1-\beta)+\frac{\kappa}{3}(\theta-5)\frac{1+\beta}{1+\kappa}\right]$

$\Xi=\frac{5}{16}\tau_t\tau_r\left[1-\alpha^2+(1-\beta^2)\left(1+\frac{1}{3}\frac{\theta-5}{1+\kappa}\right)\right]$

$\gamma_{A_t}=\frac{Z_r-Z_t-2\zeta^*}{(Y_t-2\zeta^*)(Z_r-2\zeta^*)-Y_rZ_t}$

$\gamma_{A_r}=\frac{Y_t-Y_r-2\zeta^*}{(Y_t-2\zeta^*)(Z_r-2\zeta^*)-Y_rZ_t}$

$\gamma_{B_t}=\zeta^*\frac{\gamma_{A_t}\left(Z_r-\frac{3}{2}\zeta^*\right)-\gamma_{A_r}Z_t}{\left(Y_t-\frac{3}{2}\zeta^*\right)\left(Z_r-\frac{3}{2}\zeta^*\right)-Y_rZ_t}$

$\gamma_{B_r}=\zeta^*\frac{\gamma_{A_r}\left(Y_t-\frac{3}{2}\zeta^*\right)-\gamma_{A_t}Y_r}{\left(Y_t-\frac{3}{2}\zeta^*\right)\left(Z_r-\frac{3}{2}\zeta^*\right)-Y_rZ_t}$

$Y_t=\frac{41}{12}\left(\widetilde{\alpha}+\widetilde{\beta}\right)-\frac{33}{12}\left(\widetilde{\alpha}^2+\widetilde{\beta}^2\right)-\frac{4}{3}\widetilde{\alpha}\widetilde{\beta}-\frac{7\theta}{12}\frac{\widetilde{\beta}^2}{\kappa}$

$Z_t=-\frac{5\theta}{6}\frac{\widetilde{\beta}^2}{\kappa}$

$Y_r=-\frac{25}{36}\frac{\widetilde{\beta}}{\kappa}\left(1+3\frac{\widetilde{\beta}}{\theta}-5\widetilde{\beta}\right)$

$Z_r=\frac{5}{6}\left(\widetilde{\alpha}+\widetilde{\beta}\right)+\frac{5}{18}\frac{\widetilde{\beta}}{\kappa}\left(7-3\frac{\widetilde{\beta}}{\kappa}-6\widetilde{\beta}-4\widetilde{\alpha}\right)$

$\widetilde{\alpha}=\frac{1+\alpha}{2},\ \widetilde{\beta}=\frac{1+\beta}{2}\frac{\kappa}{\kappa+1}$

$T_t^{(0)}/T=\tau_t=\frac{2}{1+\theta},\ T_r^{(0)}/T=\tau_r=\frac{2\theta}{1+\theta}$

$\theta=\sqrt{1+h^2}+h,\ h\equiv\frac{(1+\kappa)^2}{2\kappa(1+\beta)^2}\left[1-\alpha^2-(1-\beta^2)\frac{1-\kappa}{1+\kappa}\right]$

$\nu=\frac{16}{5}\sigma^2 n\sqrt{\pi\tau_t T/m}$

$\zeta^{(0)}/\nu=\zeta^*=\frac{5}{12}\frac{1}{1+\theta}\left[1-\alpha^2+(1-\beta^2)\frac{\theta+\kappa}{1+\kappa}\right]$

$\eta=(n\tau_t T/\nu)/(\nu_\eta^*-\frac{1}{2}\zeta^*)$

$\eta_b=(n\tau_t\tau_r T/\nu)\gamma_E$

$\lambda=\tau_t\lambda_t+\tau_r\lambda_r,\ \lambda_t=\frac{5}{2}(n\tau_t T/m\nu)\gamma_{A_t},\ \lambda_r=\frac{3}{2}(n\tau_t T/m\nu)\gamma_{A_r}$

$\mu=\mu_t+\mu_r,\ \mu_t=\frac{5}{2}(\tau_t^2 T^2/m\nu)\gamma_{B_t},\ \mu_r=\frac{3}{2}(\tau_t\tau_r T^2/m\nu)\gamma_{B_r}$

$\xi=\frac{1}{2}(\tau_t\xi_t+\tau_r\xi_r)=\gamma_E\Xi,\ \xi_t=\gamma_E\Xi_t,\ \xi_r=\gamma_E\Xi_r$

$\nu_\eta^*=(\widetilde{\alpha}+\widetilde{\beta})(2-\widetilde{\alpha}-\widetilde{\beta})+\widetilde{\beta}^2\theta/6\kappa$

$\gamma_E=\frac{2}{3}(\Xi_t-\Xi_r-\zeta^*)^{-1}$

$\Xi_t=\frac{5}{8}\tau_r\left[1-\alpha^2+(1-\beta^2)\frac{\kappa}{1+\kappa}-\frac{\kappa}{3}(\theta-5)\left(\frac{1+\beta}{1+\kappa}\right)^2\right]$

$\Xi_r=\frac{5}{8}\tau_t\frac{1+\beta}{1+\kappa}\left[\frac{\theta-2}{3}(1-\beta)+\frac{\kappa}{3}(\theta-5)\frac{1+\beta}{1+\kappa}\right]$

$\Xi=\frac{5}{16}\tau_t\tau_r\left[1-\alpha^2+(1-\beta^2)\left(1+\frac{1}{3}\frac{\theta-5}{1+\kappa}\right)\right]$

$\gamma_{A_t}=\frac{Z_r-Z_t-2\zeta^*}{(Y_t-2\zeta^*)(Z_r-2\zeta^*)-Y_rZ_t}$

$\gamma_{A_r}=\frac{Y_t-Y_r-2\zeta^*}{(Y_t-2\zeta^*)(Z_r-2\zeta^*)-Y_rZ_t}$

Special limiting cases

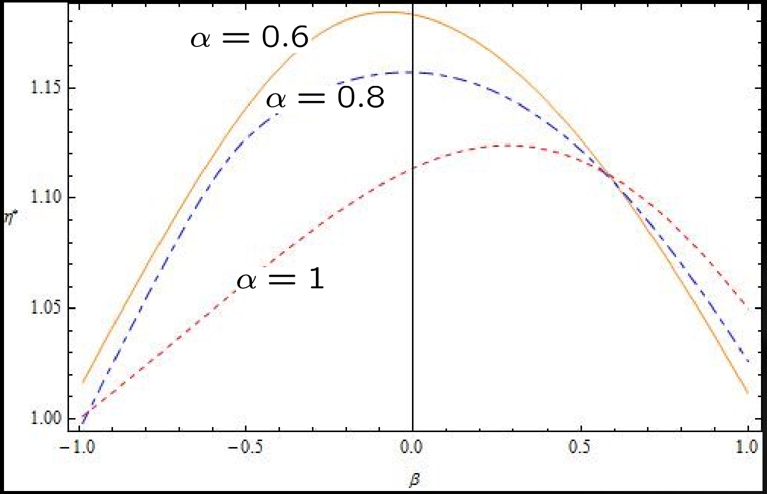
Quantity	Pure smooth	Quasi-smooth limit	Perfectly rough and elastic
	$(\beta=-1)$	$(\beta\to-1)$	$(\alpha=\beta=1)$
η^*	$\frac{24}{(1+\alpha)(13-\alpha)}$	$\frac{24}{(1+\alpha)(19-7\alpha)}$	$\frac{6(1+\kappa)^2}{6+13\kappa}$
η_b^*	0	$\frac{8}{5(1-\alpha^2)}$	$\frac{(1+\kappa)^2}{10\kappa}$
λ^*	$\frac{64}{(1+\alpha)(9+7\alpha)}$	$\frac{48}{25(1+\alpha)}$	$\frac{12(1+\kappa)^2\left(37+151\kappa+50\kappa^2\right)}{25\left(12+75\kappa+101\kappa^2+102\kappa^3\right)}$
μ^*	$\frac{1280(1-\alpha)}{(1+\alpha)(9+7\alpha)(19-3\alpha)}$	0	0
ξ	0	0	0

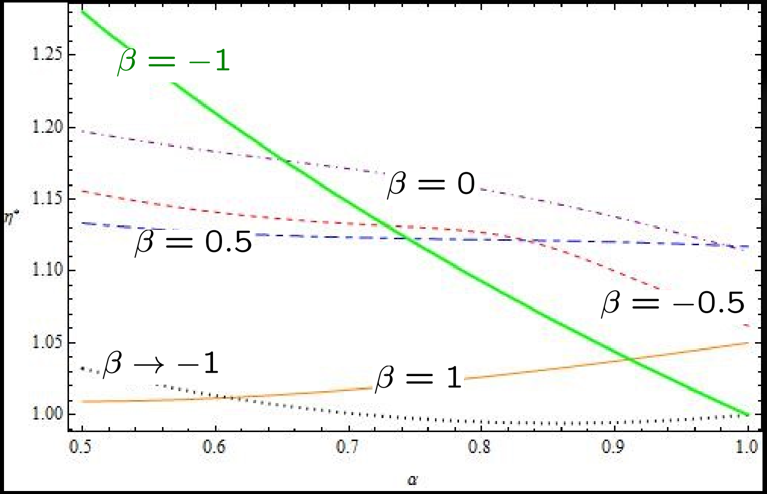
Brey, Dufty, Kim, Santos
(1998)

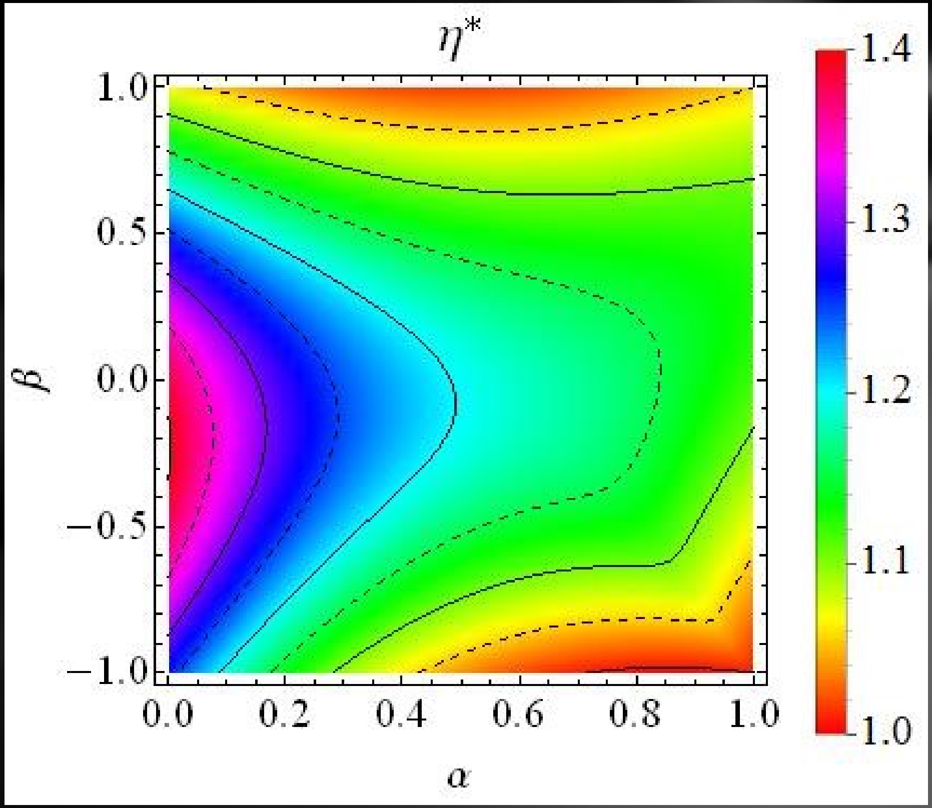
Pidduck
(1922)

Shear viscosity

$$P_{ij} = p\delta_{ij} - \eta\left(\nabla_i u_j + \nabla_j u_i - \frac{2}{d}\delta_{ij}\nabla\cdot\mathbf{u}\right) - \eta_b\delta_{ij}\nabla\cdot\mathbf{u}$$

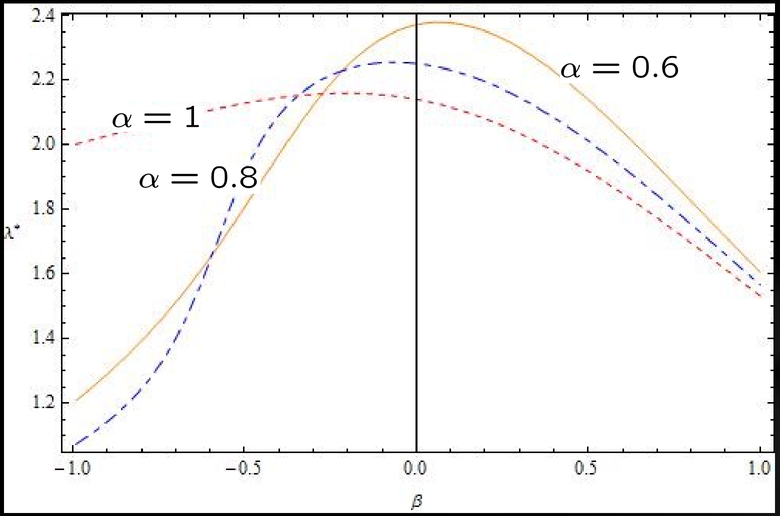


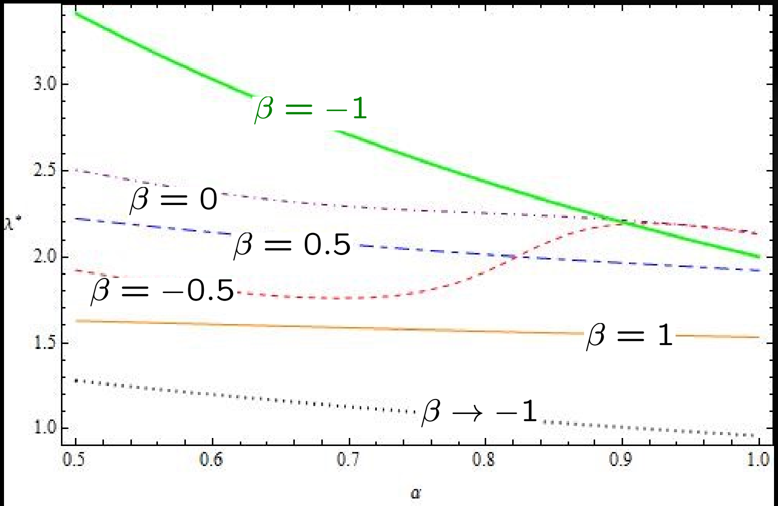


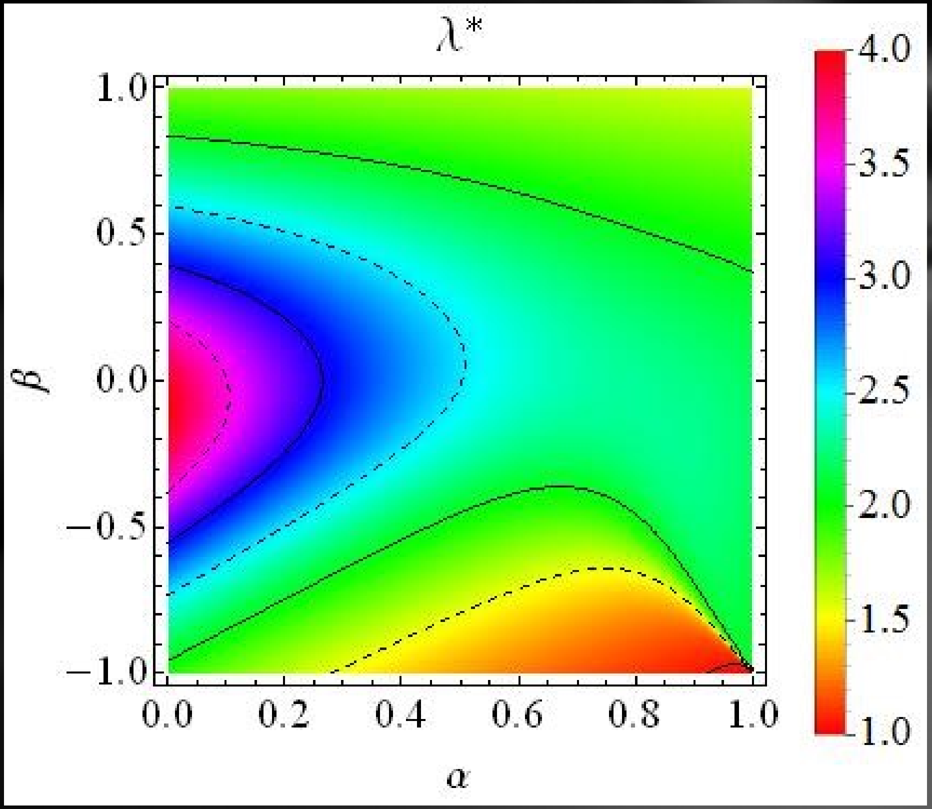


Thermal conductivity

$$\mathbf{q} = -\lambda \nabla T - \mu \nabla n$$

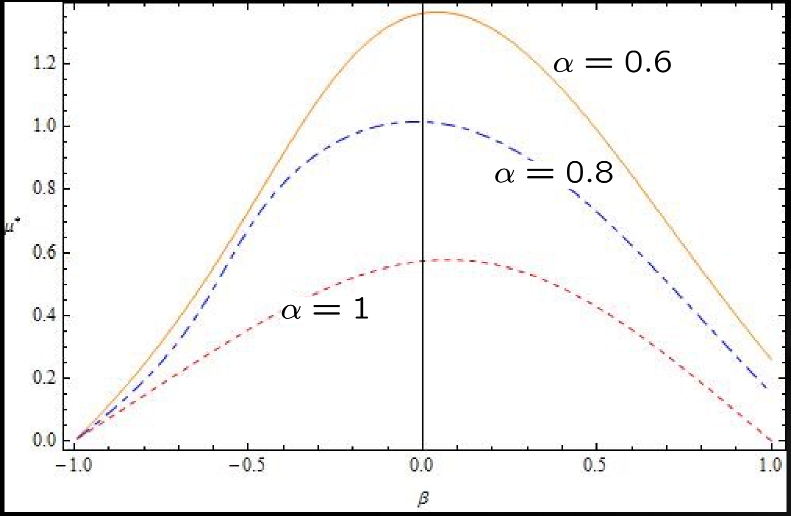


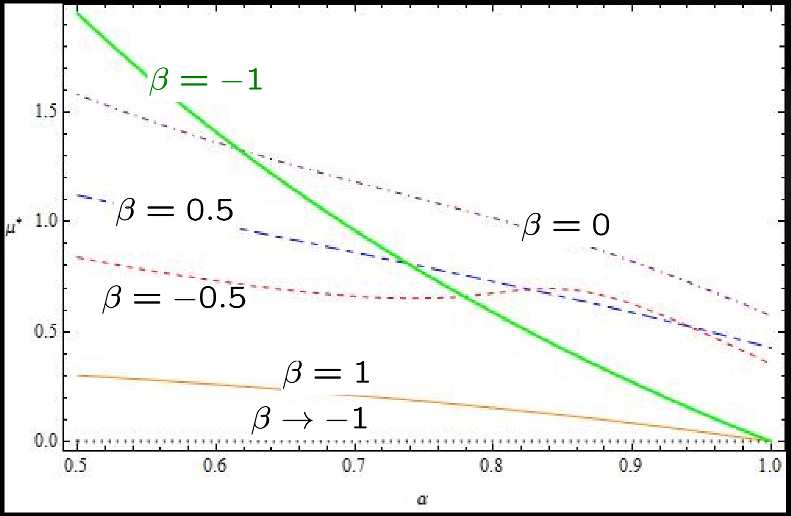


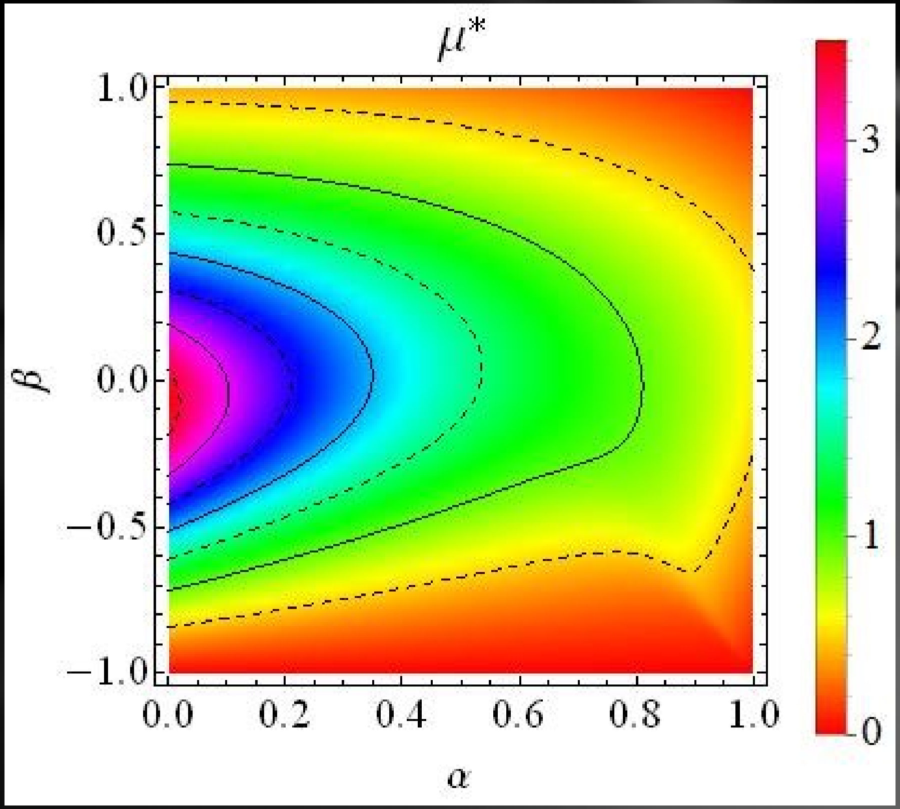


Dufour-like coefficient

$$\mathbf{q} = -\lambda \nabla T - \mu \nabla n$$

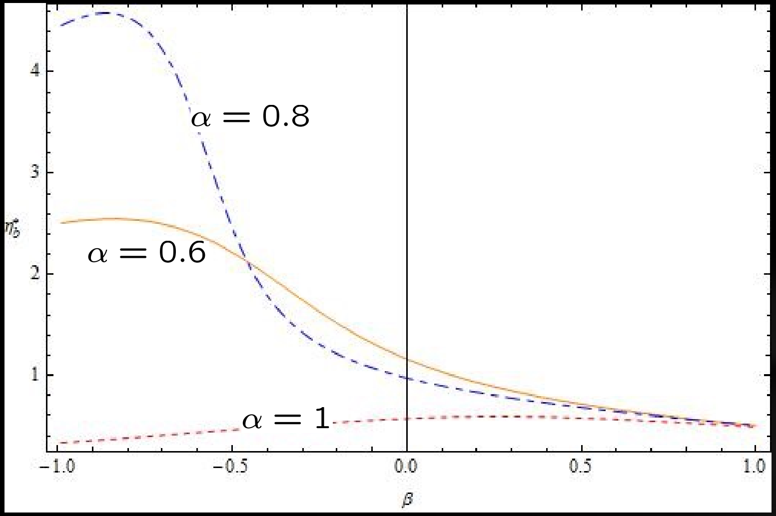


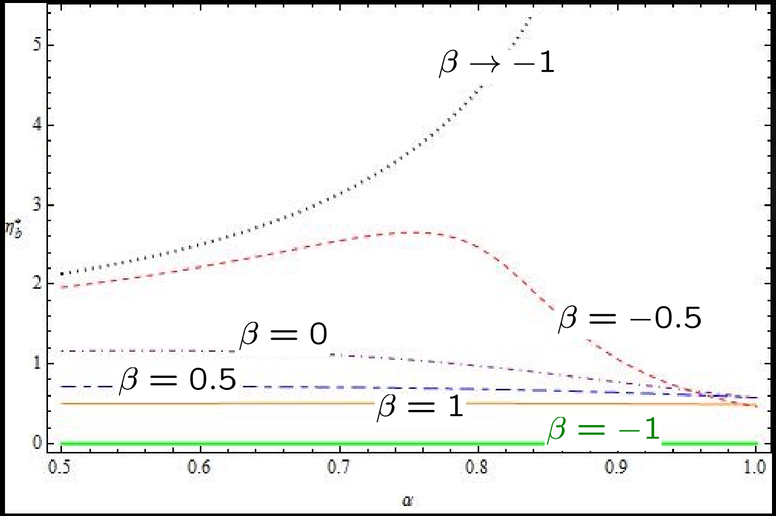


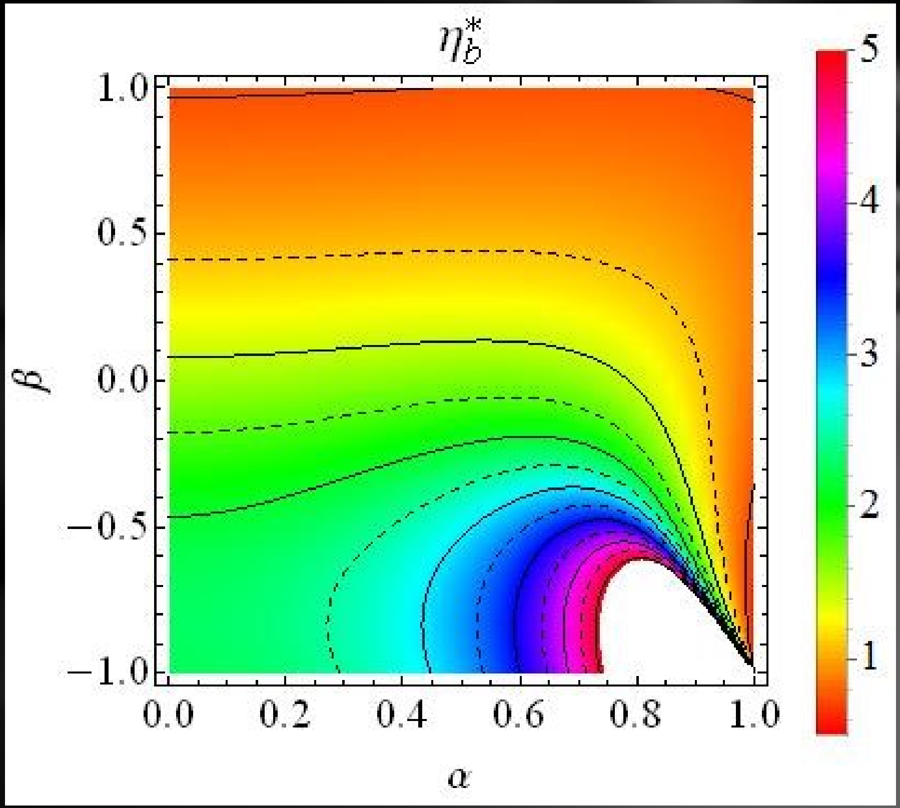


Bulk viscosity

$$P_{ij} = p\delta_{ij} - \eta\left(\nabla_i u_j + \nabla_j u_i - \frac{2}{d}\delta_{ij}\nabla\cdot\mathbf{u}\right) - \eta_b\delta_{ij}\nabla\cdot\mathbf{u}$$

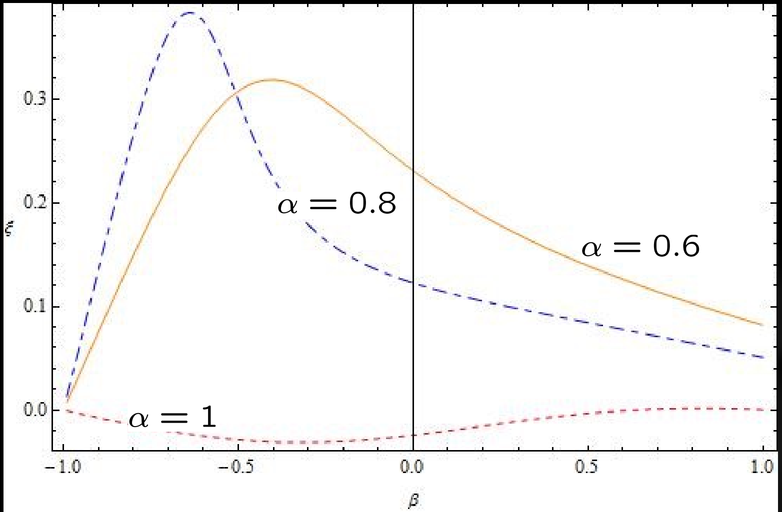


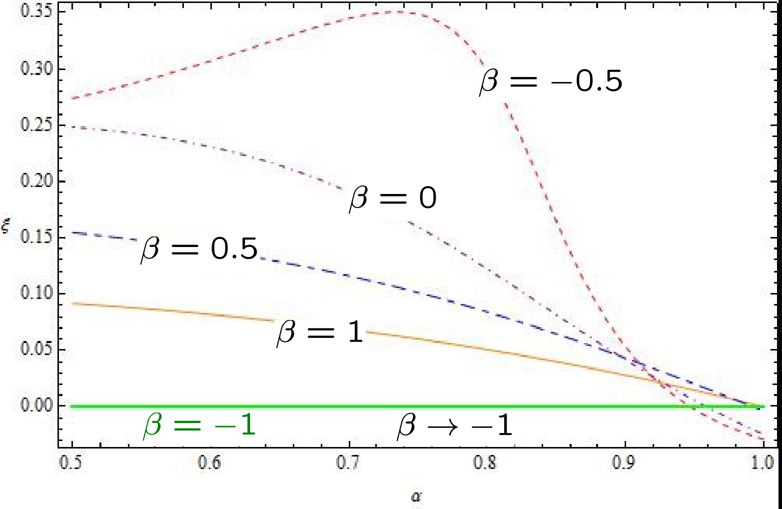


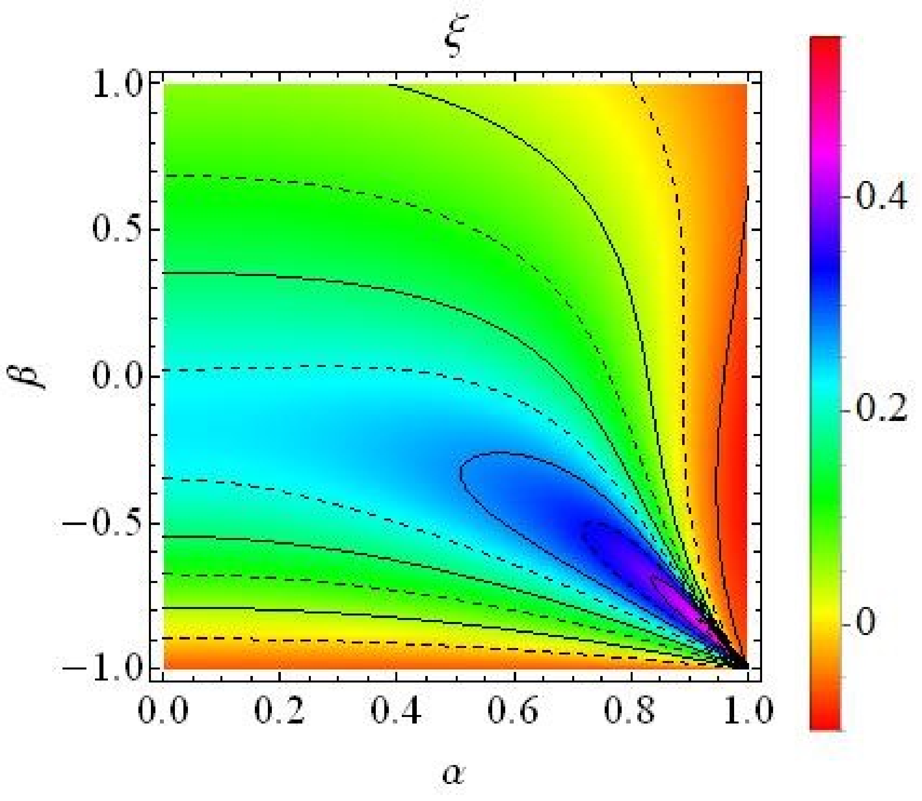


Cooling rate coefficient

$$\zeta = \zeta^{(0)} - \xi \nabla \cdot \mathbf{u}$$







Instability of the Homogeneous Cooling State

J. Fluid Mech. (2013), *vol.* 729, *pp.* 484–495. © Cambridge University Press 2013 484
doi:10.1017/jfm.2013.328

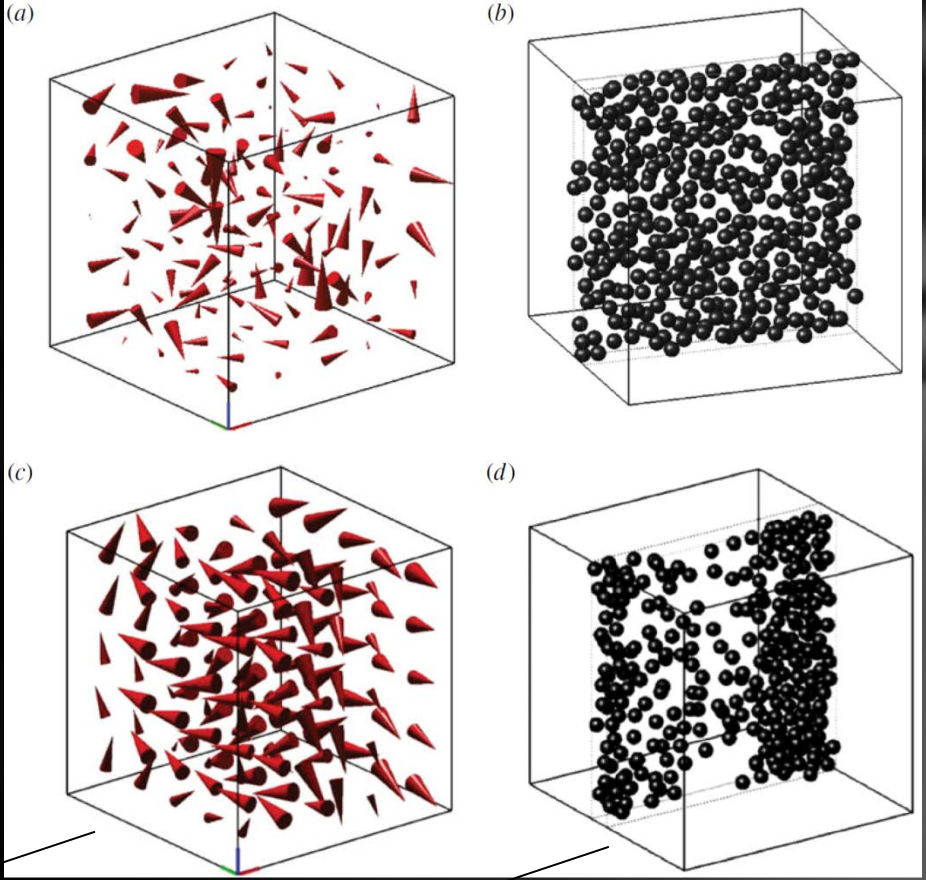
Dual role of friction in granular flows:
attenuation versus enhancement of instabilities

Peter P. Mitrano, Steven R. Dahl, Andrew M. Hilger, Christopher J. Ewasko
and Christine M. Hrenya†

$$\alpha = 0.7, \quad \beta = -0.7$$

$$\text{volume fraction } \phi = 0.3$$

$$L/\sigma = 10$$



4 collisions per particle

100 collisions per particle

Vortices

Clustering

V. Garzó, A. S., and G. M. Kremer, in preparation

Linear stability analysis

$$\left.\begin{aligned} n(\mathbf{r},t) &= n_H\left[1+\delta n^*(\mathbf{r},t)\right] \\ \mathbf{u}(\mathbf{r},t) &= \mathbf{u}_H + v_H(t)\delta\mathbf{u}^*(\mathbf{r},t) \\ T(\mathbf{r},t) &= T_H(t)\left[1+\delta T^*(\mathbf{r},t)\right] \end{aligned}\right\} \delta y_\alpha(\mathbf{r},t), \quad \alpha = 1,\ldots,5$$

Fourier-Laplace transform:

$$\delta y_\alpha(\mathbf{r},t) = \delta y_{\alpha;\mathbf{k},\omega} e^{i\mathbf{k}\cdot\mathbf{r}^*} e^{-\omega(k)s} \qquad \left[\mathbf{r}^* \equiv \frac{\mathbf{r}}{\text{m.f.p.}}, \quad s \equiv \frac{1}{2}\int_0^t dt'\, \nu_H(t')\right]$$

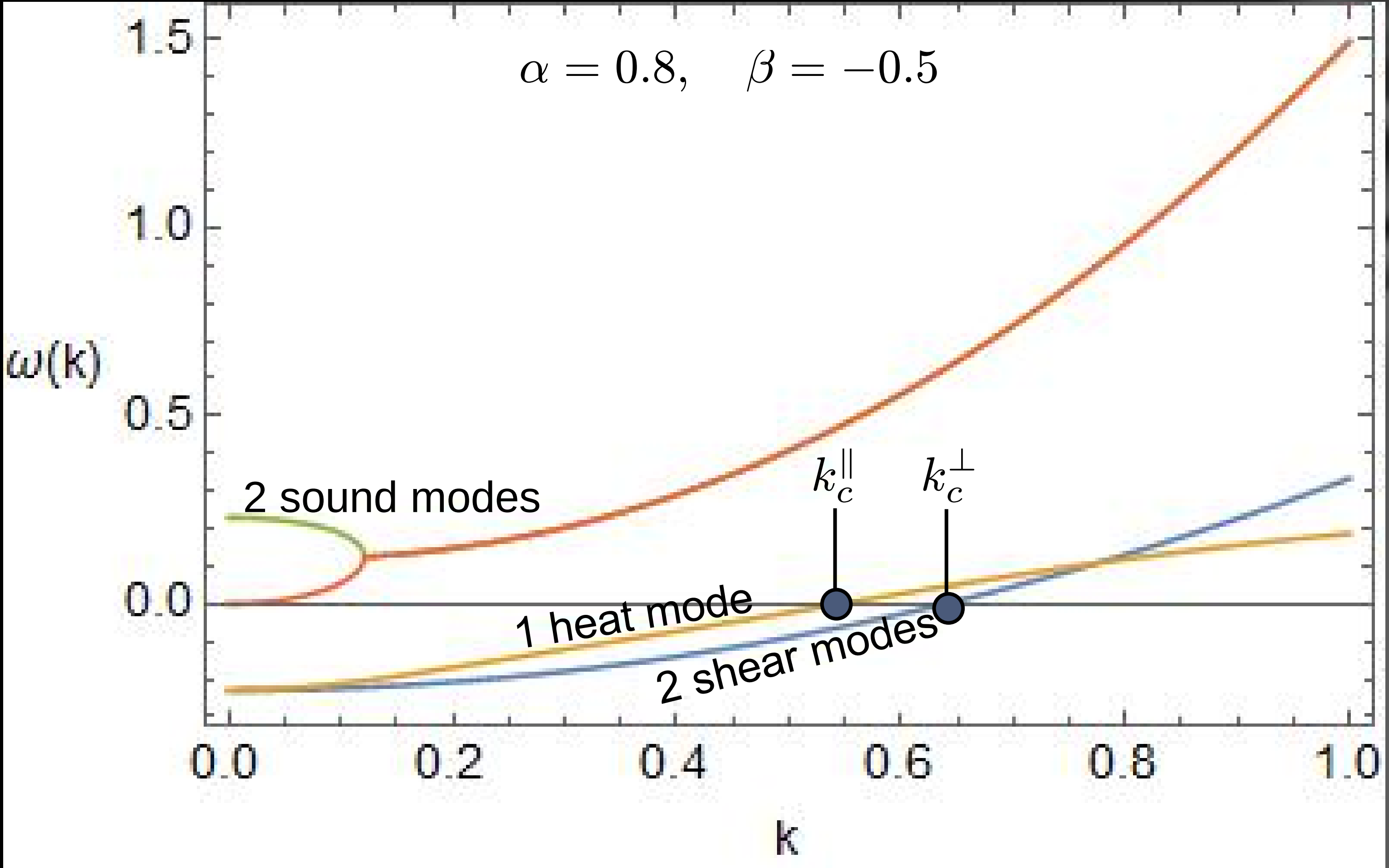
Characteristic equation:

$\det\left[\mathsf{M}(k) - \omega(k)\mathsf{I}\right] = 0 \Rightarrow$ Dispersion relation

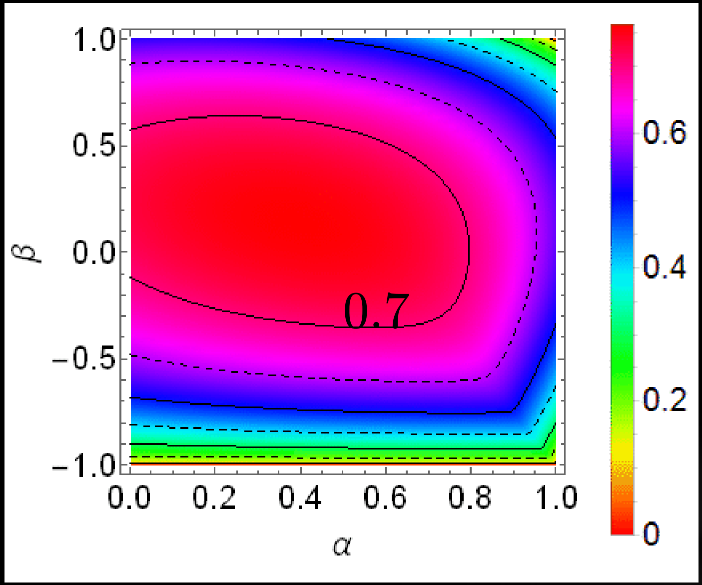
$\omega(k) < 0 \Rightarrow$ Instability

V. Garzó, A. S., and G. M. Kremer, in preparation

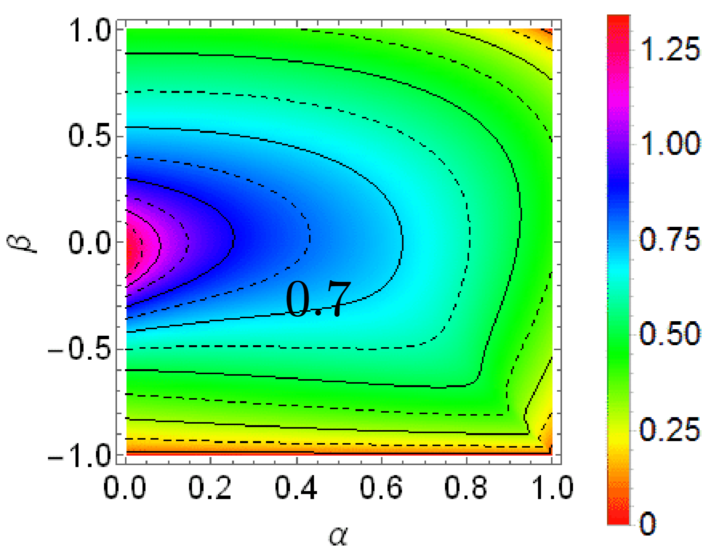
Dispersion relations



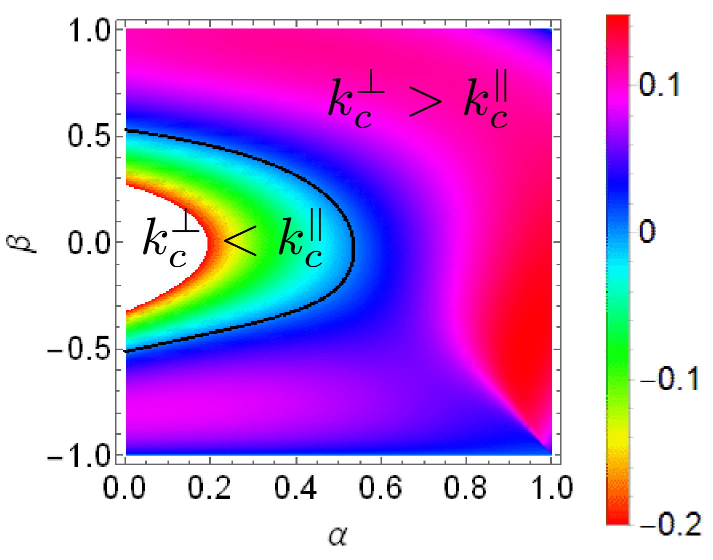
$$k_c^{\perp}(\alpha,\beta)=\sqrt{2\zeta^*/\eta^*}$$



$$k_c^{\parallel}(\alpha,\beta)=\sqrt{8\zeta^*/5(\lambda^*-\mu^*)}$$



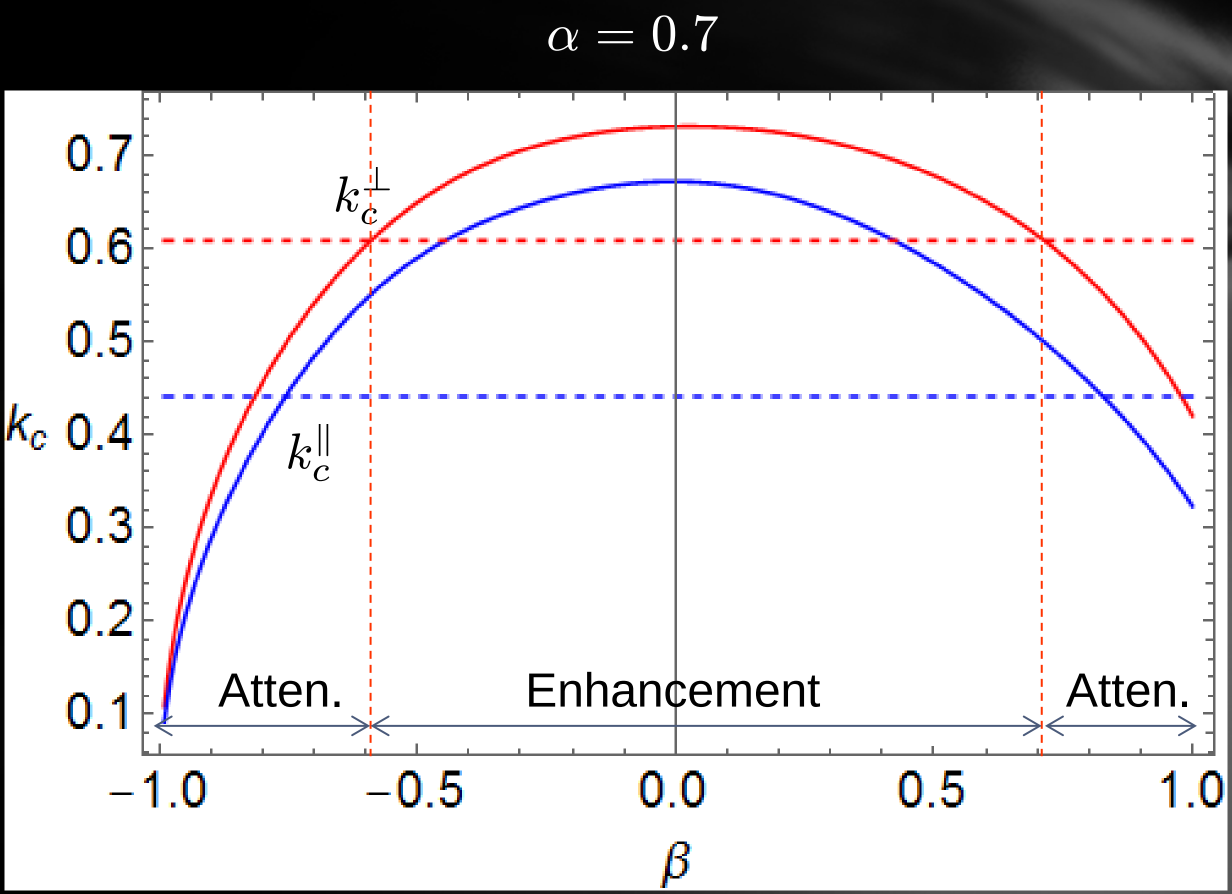
The shear modes (vortices) are more unstable than the heat mode (clusters), except for high inelasticity and medium roughness



$$k_c^{\perp}(\alpha,\beta)-k_c^{\parallel}(\alpha,\beta)$$

Comparison with the pure smooth case

Medium roughness enhances instabilities, while small and high levels of roughness attenuate it

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Dual role of friction in granular flows:
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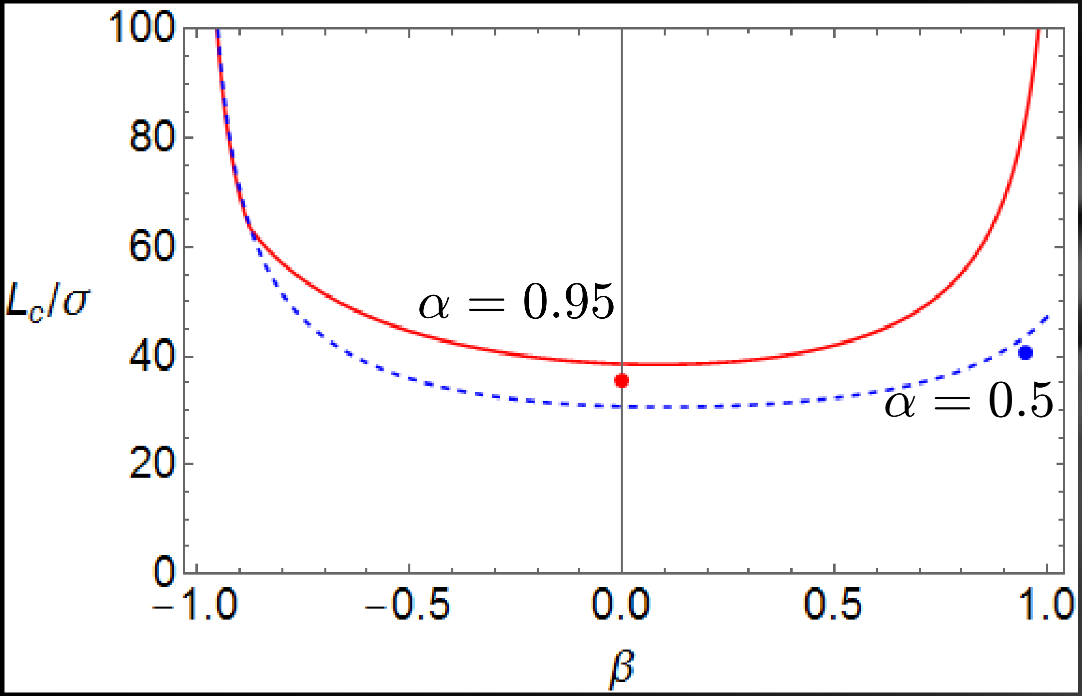
Comparison with preliminary MD simulations

volume fraction $\phi = 0.05$



(MD points, courtesy of Peter Mitrano)

Outline

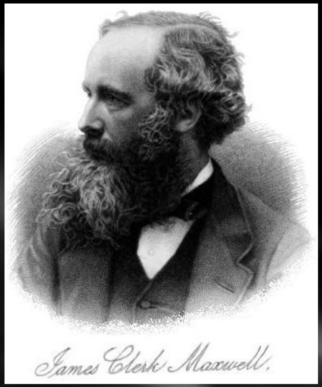
- Introduction. Granular hydrodynamics
- The inelastic rough hard-sphere model (IRHSM). Navier-Stokes coefficients
- The inelastic Maxwell model (IMM) Burnett coefficients
- Conclusions

Hydrodynamic order

Burnett	**2014 (*d*D, Exact)**	**?**	**?**
Navier-Stokes (NS)	2003 (*d*D, Exact)	1998 (Sonine appr.)	**2014 (3D, Sonine appr.)**
	Inelastic Maxwell model (IMM)	Inelastic smooth hard-sphere model (ISHSM)	Inelastic rough hard-sphere model (IRHSM)

Model complexity

Inelastic Maxwell model (IMM)



(1831–1879)

- First proposed by Bobylev, Carrillo & Gamba (2000), Ben-Naim & Krapivsky (2000), Bobylev & Cercignani (2002), Ernst & Brito (2002), …
- The hard-sphere collision rate (proportional to the *relative velocity*) is replaced by an effective (mean-field) *constant* value.
- Otherwise, the collision rule remains unchanged.

Inelastic Maxwell model (IMM)

Boltzmann eq.:

$$\partial_t f(\mathbf{r},\mathbf{v},t) \;+\; \mathbf{v}\cdot\nabla f(\mathbf{r},\mathbf{v},t) = J[\mathbf{r},\mathbf{v},t|f]$$

ISHSM:

$$J\left[\mathbf{v}_1|f\right] \;=\; \sigma^{d-1}\int \mathrm{d}\mathbf{v}_2 \int \mathrm{d}\widehat{\boldsymbol{\sigma}}\,\Theta\left(\mathbf{g}\cdot\widehat{\boldsymbol{\sigma}}\right)\boxed{\left(\mathbf{g}\cdot\widehat{\boldsymbol{\sigma}}\right)}\left[\alpha^{-2}f(\mathbf{v}_1')f(\mathbf{v}_2') - f(\mathbf{v}_1)f(\mathbf{v}_2)\right]$$

IMM:

$$J\left[\mathbf{v}_1|f\right] \;=\; \frac{d+2}{2\Omega_d}\,\frac{\boxed{\nu}}{n}\int \mathrm{d}\mathbf{v}_2 \int \mathrm{d}\widehat{\boldsymbol{\sigma}}\,\left[\alpha^{-1}f(\mathbf{v}_1')f(\mathbf{v}_2') - f(\mathbf{v}_1)f(\mathbf{v}_2)\right],$$

$$\nu \;\propto\; T^{\gamma}, \qquad \gamma = \frac{1}{2}$$

Exact moments

$$\begin{aligned}
\text{Collisional moment of order } k &= \text{Linear combination of velocity moments} \\
&\quad \text{of order equal to or smaller than } k \\
m\int d\mathbf{v}\, V_i V_j J[\mathbf{v},f] &= -\nu_{0|2}(\alpha)\left(P_{ij}-p\delta_{ij}\right)-\zeta(\alpha)p\delta_{ij} \\
\frac{m}{2}\int d\mathbf{v}\, V^2 V_i J[\mathbf{v},f] &= -\nu_{2|1}(\alpha)q_i \\
\cdots &= \cdots
\end{aligned}$$

N. Khalil, V. Garzó, and A. S., Phys. Rev. E **89**, 052201 (2014)

Results

$$
\begin{aligned}
P_{ij}^{(2)} =& a_1 \frac{\lambda_0}{\nu}\left(\nabla_i \nabla_j T - \frac{1}{d}\delta_{ij}\nabla^2 T\right) + a_2 \frac{T\lambda_0}{p\nu}\left(\nabla_i \nabla_j p - + \frac{1}{d}\delta_{ij}\nabla^2 p\right) \\
& a_3 \frac{\lambda_0}{T\nu}\left[\nabla_i T \nabla_j T - \frac{1}{d}\delta_{ij}(\nabla T)^2\right] + a_4 \frac{T\lambda_0}{p^2\nu}\left[\nabla_i p \nabla_j p - \frac{1}{d}\delta_{ij}(\nabla p)^2\right] \\
& + a_5 \frac{\lambda_0}{p\nu}\left(\nabla_i T \nabla_j p + \nabla_i p \nabla_j T - \frac{2}{d}\delta_{ij}\nabla p \cdot \nabla T\right) + a_6 \frac{\eta_0}{\nu} D \left(D_{ij} - \frac{1}{d}\delta_{ij} D\right) \\
& + a_7 \frac{\eta_0}{\nu}\left[D_{ik}D_{kj} - \omega_{ik}\omega_{kj} - \frac{1}{d}\delta_{ij}\left(D_{lk}D_{kl} - \omega_{lk}\omega_{kl}\right) + \omega_{ij}D_{kj} - D_{ik}\omega_{kj}\right] \\
q_i^{(2)} =& b_1 \frac{T\lambda_0}{\nu}\nabla^2 u_i + b_2 \frac{T\lambda_0}{\nu}\nabla_i D + b_3 \frac{\lambda_0}{\nu} D_{ij}\nabla_j T + b_4 \frac{\eta_0}{\rho\nu} D_{ij}\nabla_j p + b_5 \frac{\lambda_0}{\nu}\omega_{ij}\nabla_j T \\
& + b_6 \frac{\eta_0}{\rho\nu}\omega_{ij}\nabla_j p + b_7 \frac{\lambda_0}{\nu} D \nabla_i T + b_8 \frac{\eta_0}{\rho\nu} D \nabla_i p,
\end{aligned}
$$

$$
D \equiv \nabla \cdot \mathbf{u}, \qquad D_{ij} \equiv \frac{1}{2}(\nabla_i u_j + \nabla_j u_i), \qquad \omega_{ij} \equiv \frac{1}{2}(\nabla_j u_i - \nabla_i u_j)
$$

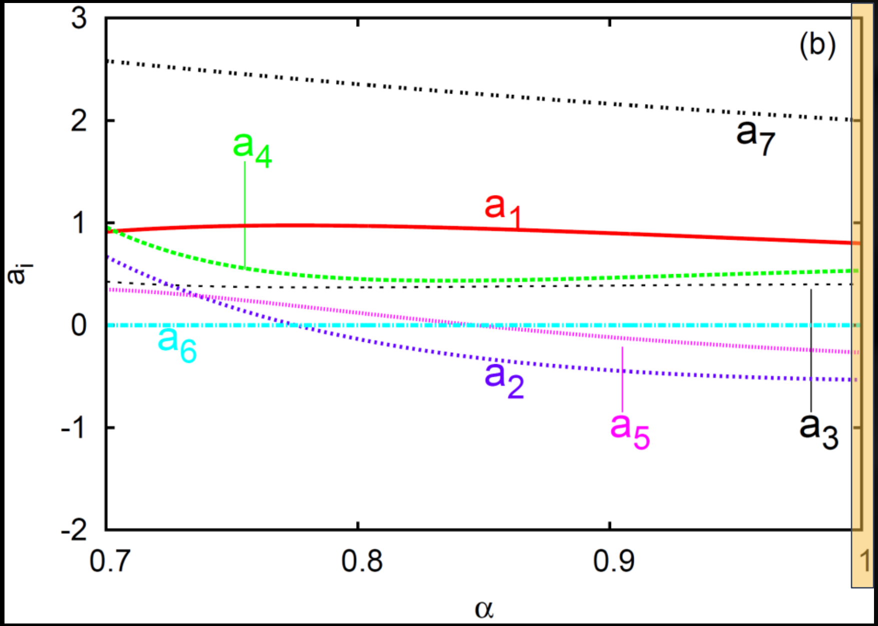
Elastic limit ($\alpha \to 1$)

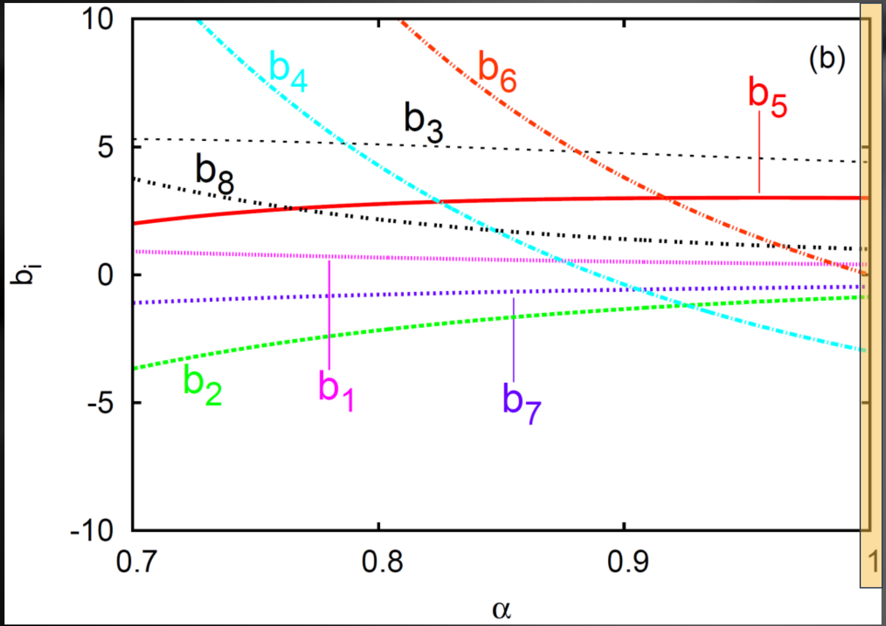
$$
\begin{array}{llll}
a_1 \to \frac{4}{d+2}, & a_2 \to -\frac{4(d-1)}{d(d+2)}, & a_3 \to \frac{4(1-\gamma)}{d+2}, & a_4 \to \frac{4(d-1)}{d(d+2)}, \\
a_5 \to -\frac{2(d-1)}{d(d+2)}, & a_6 \to \frac{2(d-4+2\gamma)}{d}, & a_7 \to 2, & \\
b_1 \to \frac{2}{d+2}, & b_2 \to -\frac{2(5d-2)}{d(d-1)(d+2)}, & b_3 \to \frac{2\left[d^2+7d-6-2(d-1)\gamma\right]}{(d-1)(d+2)}, & b_4 \to -\frac{2d}{d-1}, \\
b_5 \to \frac{2d}{d-1}, & b_6 \to 0, & b_7 \to \frac{d^3-2d^2-18d+12+2(d^2+4d-2)\gamma}{d(d-1)(d+2)}, & b_8 \to \frac{2}{d-1}
\end{array}
$$

- Exact results for Maxwell molecules if $\gamma = 0$.
- First Sonine approximation for any potential if $\gamma = 1 - \partial \ln \eta_0(T)/\partial \ln T$.
- Generalization to any d of Chapman & Cowling's classical expressions ($d = 3$).

Influence of inelasticity (IMM, exact)

$$d = 3, \gamma = \frac{1}{2}$$





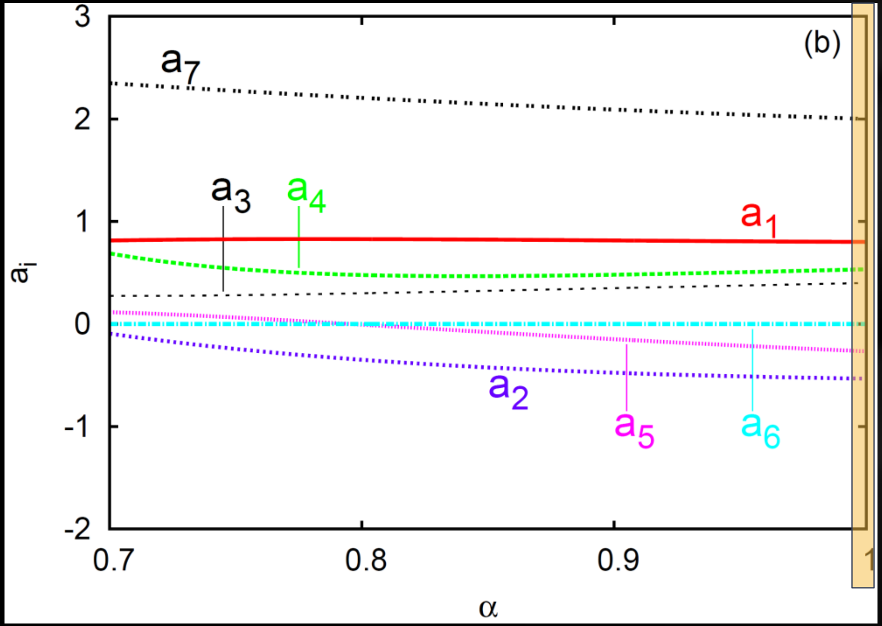
Hydrodynamic order

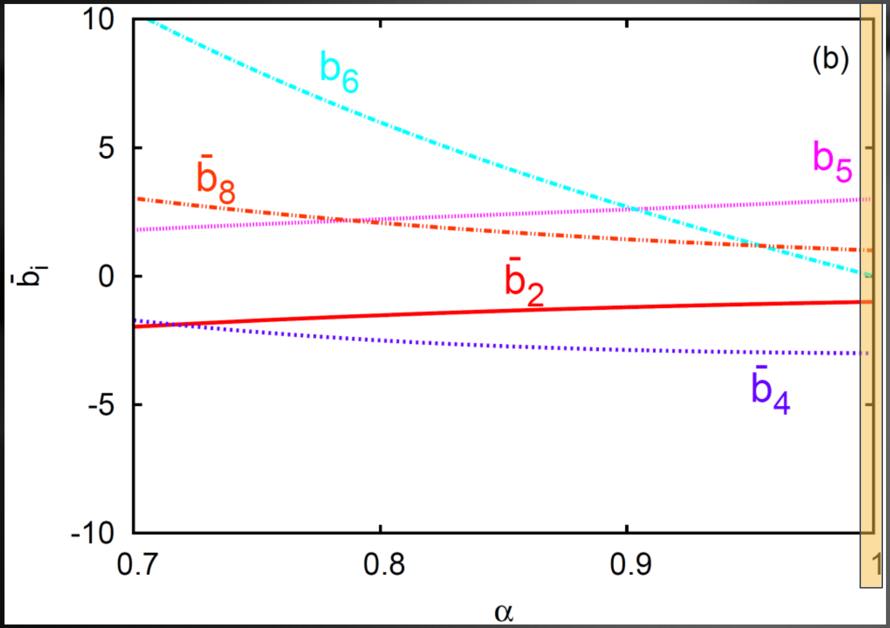
Burnett	**2014 (*d*D, Exact)**	?	**?**
Navier-Stokes (NS)	2003 (*d*D, Exact)	1998 (Sonine appr.)	**2014 (3D, Sonine appr.)**
	Inelastic Maxwell model (IMM)	Inelastic smooth hard-sphere model (ISHSM)	Inelastic rough hard-sphere model (IRHSM)

Model complexity

Influence of inelasticity (ISHSM, estimated)

$$d = 3$$





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THE TAKE-HOME MESSAGE

- **IRHSM**: Roughness (and inelasticity) have a large impact on the NS transport coefficients.
- **IMM**: *Exact* results for the Burnett coefficients can be obtained. They can be used to estimate the coefficients for the ISHSM.


THANK
YOU!


What people think about during your conference talk
Hee hee! Animated arrows!
Typo on slide 14, line 3, centre-left. Noted.
Hey! That's what I'm working on!
(RANDOM HAPPY THOUGHTS]
I'm up next I'm up next I'm up next
I'm in the wrong session.
©The Upturned Microscope

Origin of the singular behavior in the quasi-smooth limit